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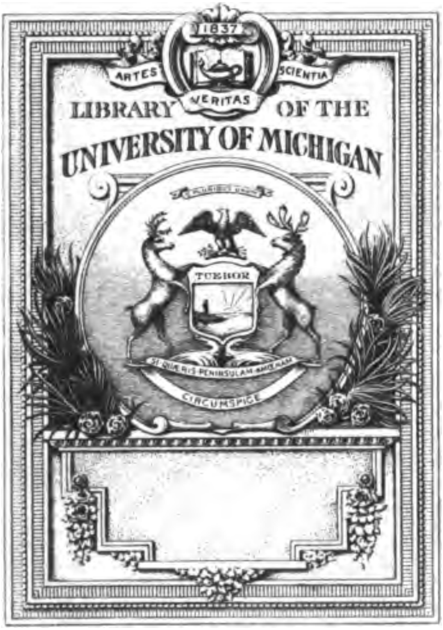

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THE

ELEMENTS

OF

PLANE TRIGONOMETRY.

BY

enry athan

H. N. WHEELER, A.M.

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PREFACE.

THIS book originated in a desire to give to my private pupils a better idea of the trigonometric functions of obtuse angles than they could obtain from any book with which I was acquainted.

I believe that a student can get a comprehensive and thorough knowledge of trigonometry most quickly and easily, if, at the outset, such definitions are given to the trigonometric functions as will apply to all angles: with this idea for a basis, I have endeavored to prepare an elementary text-book for general use. By beginning with an explanation of the use of the negative sign as applied to lines and angles, and then giving general definitions to the trigonometric functions, I have been able to demonstrate all the fundamental formulas in a perfectly general yet simple manner.

While I have tried to present the subject from an elementary point of view, I have not lost sight of the fact, that, to most students, trigonometry is merely a stepping-stone to something higher; and for this reason I have also tried to present the results in such a light as will make them effective tools for the student in his future work.

During the preparation and printing of the book, Prof. J. M. Peirce has kindly given me the benefit of numerous suggestions and criticisms, which have added greatly to its value. Appendix I. is entirely due to him.

Most of the examples contained in the book have been selected from other trigonometries, especially those of Todhunter, Snowball, and Hamblin Smith.

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PLANE TRIGONOMETRY.

CHAPTER I.

DEFINITIONS AND FIRST PRINCIPLES.

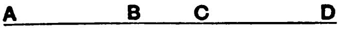
§ 1. TRIGONOMETRY, as the name itself indicates, was originally simply a method of *measuring triangles* by numerical computation from any sufficient data. This method involves the use of certain functions of the angles, which we call the *trigonometric functions.* Modern trigonometry embraces primarily the complete theory of these functions in their most general relations; while the measurement of triangles, or trigonometry proper, is reduced to a mere branch, or application, of the theory.

That part of trigonometry which is occupied with the investigation of the general properties of the trigonometric functions is called by some writers *Angular Analysis*, or *Goniometry;* but it is to be observed that these properties, as well as the very conception of the trigonometric functions themselves, are derived from the doctrine of similar triangles, and the whole subject (at least in its elements) may be regarded as a development of that doctrine.

We must begin our exposition of trigonometry by treating of certain principles adopted in the measurement of lines and angles, the two kinds of geometric quantity which will enter into our formulas.

§ 2. *Measurement of lines.* — A line may be regarded as traced by the movement of a point. Thus we may regard the straight line AD (Fig. 1) as having been traced by the movement of a point from A, the beginning of the line, to D, the end of the line.

FIG. 1.



Straight lines may be added by placing the beginning of the second upon the end of the first (so as to form a continuous straight line), the beginning of the third upon the end of the second, and so on. The straight line contained between the beginning of the first and the end of the last is a line equal to the sum of the lines to be added. Thus (Fig. 1) $AD = AB + BC + CD$.

§ 3. *a.* There are many problems involving straight lines, in which it is neither necessary nor profitable to take into consideration any thing except the lengths of these lines. There are, on the other hand, many cases where it will be useful to take into account direction as well as length.

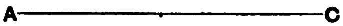
b. If we define any arbitrary direction to be positive, then the opposite direction will be negative. For, if AB is any line, —AB can only be that line which added to AB, by the method of § 2, produces 0; and that is the same line as AB taken in the opposite direction.

The length of a straight line has no algebraic sign: it is always the same, whatever be the position or direction of the line.

c. The sign + or — prefixed to a number or letter denoting a line has reference merely to direction. Thus, if a is positive, the line $+a$ is a line whose direction is positive, and whose length is a units; and the line $-a$ is a line whose direction is negative, and whose length is, as before, a units. But, if a is negative, $+a$ is a line whose direction is negative, and $-a$ is a line of the same length, whose direction is positive. The terms "positive" and "negative" are used merely to enable us to distinguish between two opposite directions; and we can in any figure assume *any directions* (not opposite) to be positive, but, having once assumed certain directions to be positive, the opposite directions must be negative. (*v.*, however, § 20, *d.*)

d. It is possible to represent a straight line completely by an expression involving the points at its extremities. Thus AC (Fig. 2) is an expression for the straight line which joins the points A and C, whose length is the distance between the points, and whose direction is the direction from A, the point indicated by the first letter, to C, the point indicated by the second letter. $-$AC denotes a line having the same length as the above, but an opposite direction, but this latter line is the line CA. $\therefore CA = -AC$, or $AC + CA = 0$. The truth of this equation is evident if, as in § 2, we regard a line as traced by the movement of a point. For if a point moves from A to C, and then from C back to A, it is evident that its final distance from the starting-point will be nothing.

FIG. 2.



Similarly in each of the figures 3, 4, and 5, we have, —

$$AB + BC + CA = 0$$

$$BC + CA + AB = 0$$

$$CB + BA + AC = 0$$

FIG. 3.

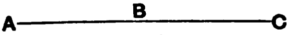


FIG. 4.

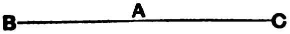
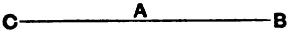


FIG. 5.



This principle may be extended to any number of points in the same straight line.

§ **4.** *Measurement of angles.*—A right angle is divided into 90 equal parts called degrees; a degree is subdivided into 60 minutes; and a minute, into 60 seconds. The degree, minute, and second are distinguished by the characters ° ′ ″ respectively. A degree of arc is $\frac{1}{360}$ of the circumference of the circle to which the arc belongs. It evidently has different lengths for different circles. The degree of arc is subdivided in the same manner as the degree of angle. $\frac{1}{4}$ the circumference, or the arc intercepted by the sides of a right angle whose vertex is at the centre, is an arc of 90°, and is called a *quadrant.* The same term is applied to the sector bounded by two radii at right angles to each other. The four divisions of a plane separated by two lines perpendicular to each other are also occasionally called quadrants.

Central angles contain the same number of degrees as their intercepted arcs.

§ **5.** The *complement* of an angle or arc is the remainder obtained by subtracting the angle or arc from 90°.

The *supplement* of an angle or arc is the remainder obtained by subtracting the angle or arc from 180°.

Thus the complement of 25° is 65°; the supplement of 25° is 155°.

Two angles or arcs are therefore complements of each other when their sum is 90°; and they are supplements of each other when their sum is 180°.

According to these definitions, the complement of an angle or arc that exceeds 90° is negative. Thus the complement of 100° is $90° - 100° = -10°$. In like manner the supplement of 200° is $180° - 200° = -20°$.

§ 6. *Circular measure of an angle.*

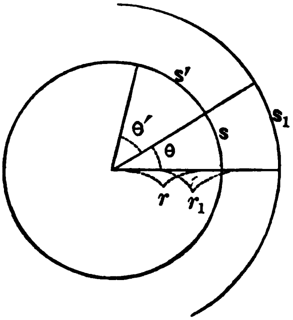
As has been indicated above, an angle may be measured in degrees and subdivisions of a degree. There is, however, another method of measurement in use, called the *Circular Method*, by which an angle is measured by the ratio of its arc in the circle described from its vertex as a centre to the radius of that circle. To show that this method is legitimate, it must be proved that *any two angles are proportional to the ratios by which we propose to measure them.*

Now, in the same circle, or equal circles, angles at the centre are proportional to their intercepted arcs. Therefore (Fig. 6), —

$$\theta : \theta' = s : s' \qquad \text{(I.)}$$

$$\therefore \theta : \theta' = \frac{s}{r} : \frac{s'}{r} \qquad \text{(II.)}$$

FIG. 6.



But, since similar arcs are proportional to their radii, we have $\frac{s}{r} = \frac{s_1}{r_1}$; that is, the ratio of *intercepted arc to radius* is the same for the same angle, whatever the value of the radius. Substituting this value of $\frac{s}{r}$ in equation II. above, we get $\theta : \theta' = \frac{s_1}{r_1} : \frac{s'}{r}$. *Q.E.D.*

§ 7. COROLLARY. Any quantity can be put equal to its measure. We can therefore put $\theta = \frac{s}{r}$. Now when the radius (r) is unity, we have $\theta = \frac{s}{1} = s$; therefore the circular measure of an angle is equal to *the length of the intercepted arc in a circle whose radius is unity.*

§ 8. *Angular unit of circular measure.*

As in § 7 we can put $\theta = \frac{s}{r}$. Making $s = r$ we shall have $\theta = \frac{r}{r} = 1$. Therefore the angular unit of circular measure is *that angle whose intercepted arc is equal to the radius.*

§ 9. *To change from circular measure to degree measure, and the converse.*

We will first find how many degrees there are in the angular unit of circular measure; i.e., we will reduce θ to degrees when $s = r$. The ratio of *circumference* to *diameter* is constant for all circles, and is approximately equal to 3.1415927, which is represented by the Greek letter π (*v.* Plane Geometry). Therefore letting C, D, and R, represent circumference, diameter, and radius respectively, we have $\frac{\mathrm{C}}{\mathrm{D}} = \pi$; $\therefore \mathrm{C} = \pi \mathrm{D}$; but $\mathrm{D} = 2\,\mathrm{R}$; $\therefore \mathrm{C} =$ arc $360^\circ = 2\,\pi\,\mathrm{R}$. In the same circle, or equal circles, central angles are proportional to their intercepted arcs: —

$$\therefore \frac{\text{angle } \theta}{\text{angle } 360^\circ} = \frac{\text{arc S}}{\text{arc C}} = \frac{\mathrm{S}}{2\pi \mathrm{R}} = (\text{when } \mathrm{S} = \mathrm{R})\, \frac{\mathrm{R}}{2\pi \mathrm{R}} = \frac{1}{2\pi}$$

$$\therefore \text{angular unit} = \frac{360^\circ}{2\pi} = \frac{180^\circ}{\pi} = \frac{180^\circ}{3.1415927}$$

$$= 57^\circ 2957795.$$

Conversely, the units of degree-measure have the following values in circular measure; and these values are also the lengths of the corresponding arcs of the unit circle: —

$$\text{circ. meas. } 1^\circ = \frac{\pi}{180} = 0.01745329,$$

$$\text{circ. meas. } 1' = \frac{1^\circ}{60} = 0.00029089,$$

$$\text{circ. meas. } 1'' = \frac{1'}{60} = 0.00000485.$$

To convert an angle from one system to the other, we have only to multiply the given measure of the angle by the value of its unit in the new system. For example, if x is the circular measure of an angle, its value in degrees is

$$x \times 57.296\ldots\ldots;$$

and the circular measure of x° is

$$x \times 0.01745\ldots;$$

§ **10.** Angle $360^\circ = \frac{\text{arc } 360^\circ}{R} = \frac{2\pi R}{R} = 2\pi$,

Angle $180^\circ = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ \pi$,

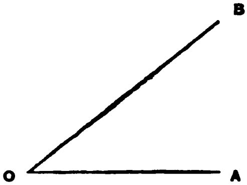
Angle $90^\circ = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ \frac{1}{2}\pi$,

Angle $45^\circ = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ \frac{1}{4}\pi$.

The expressions 2π, π, $\frac{1}{2}\pi$, and $\frac{1}{4}\pi$ are often used to denote the angles which they measure.

§ **11.** *General Angular Magnitude.* "A clear notion of the magnitude of an angle will be obtained by supposing that one of its sides as OB (Fig. 7) was at first coincident with the other side OA, and that it has revolved about the point O (turning upon O as the leg of a pair of dividers turns upon its hinge) until it has arrived at the position OB." — *Chauvenet's Geometry.*

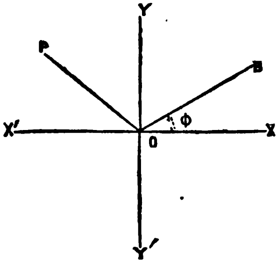
FIG. 7.



We can, then, regard an angle as the measure of the amount of the rotation of a line which turns in a plane about one of its own points. The angle AOB (Fig. 7), for example, shows how much the line OB has rotated from its original direction OA about O as a centre.

§ **12.** Since there is no limit to the rotation of a line about one of its points, there is no limit to angular magnitude.

FIG. 8.

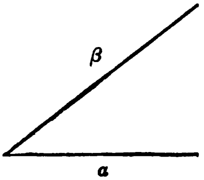


Let XX′ and YY′ (Fig. 8) be perpendicular to each other at the point O. Let OX be the original position of OB. When OB has moved to the position OY, it makes with OX an angle of 90°; when to the position OX′, an angle of 180°; when to the position OY′, an angle of 270°; when to OX again, an angle of 360°; if it continues to move to OY, an angle of 360° + 90°, or 450°; and, as long as OB continues to move, the angle XOB continues to increase. OX, the original position of the line OB, is called the *initial* line with respect to the angle XOB, while OB is called the *terminal* line with respect to this angle.

§ **13.** When we represent an angle by three letters, we shall always denote the vertex by the middle letter, a point through which the initial line passes by the first letter, and a point through which the terminal line passes by the third letter.

Prof. Peirce has made use of the following angular notation: Let α and β (Fig. 9) denote two lines. The angle which β makes with α, or the angle *alpha-beta*, as it is called, is denoted by the expression $\overset{\beta}{\alpha}$, the lower letter denoting the initial line, and the upper letter the ter-

FIG. 9.



minal line. Such symbols must be carefully distinguished from fractions. This notation may be used even in the case of lines which do not intersect.

§ 14. Rotation as considered in § 12 (i.e., opposite that of the hands of a clock) is called positive rotation. Rotation in the contrary direction is therefore negative rotation. An angle is positive or negative according as it is conceived to be generated by positive or negative rotation. Thus, we may take BOX $= -$ XOB, $\frac{\alpha}{\beta} = -\frac{\beta}{\alpha}$.

§ 15. Let AA′ and B′B″ (Figs. 10–13) be two straight lines perpendicular to each other at O, the centre of the circle of which OA is the radius. The arcs AB′, B′A′, A′B″,

FIG. 10.

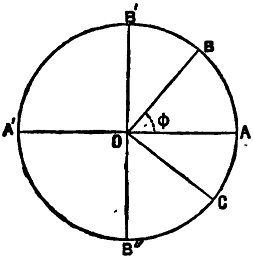


FIG. 11.

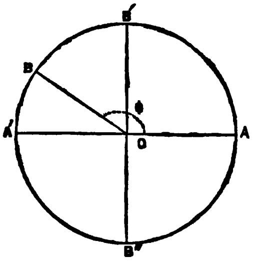


FIG. 12.

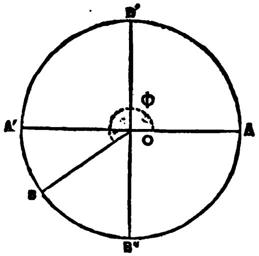
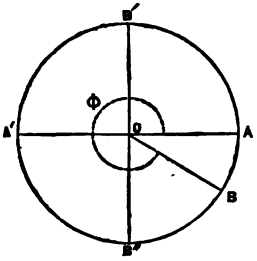


FIG. 13.



and B″A, or the sectors AOB′, B′OA′, A′OB″, and B″OA, are quadrants (*v.* § 4). In accordance with what has been said about positive rotation (*v.* § 14), we will call the sector AOB′ (or the arc AB′) the first quadrant, B′OA′ the second quadrant, A′OB″ the third quadrant, and B″OA the fourth quadrant. Therefore the angle marked φ in the figures 10, 11, 12, and 13, is an angle of (i.e. terminating in) the 1st, 2d, 3d, and 4th quadrants, respectively.

§ **16.** Angles in the same plane may be added by first placing their vertices at the same point, and then placing the initial line of the second upon the terminal line of the first, the initial line of the third upon the terminal line of the second, and so on. The angle made by the terminal line of the last with the initial line of the first is an angle equal to the sum of the angles to be added. Thus (Fig. 8) XOP = XOB + BOY + YOP. On the same principle, if α, β, γ, δ, &c., are any lines in one plane, whether intersecting at one point or not, we have: —

$${}^{\beta}\!/_{\alpha} + {}^{\gamma}\!/_{\beta} + {}^{\delta}_{\gamma} = {}^{\delta}_{\alpha}.$$

§ **17.** The angle AOB (Figs. 12, 13) may have been formed either by the positive or negative rotation of OB; we can therefore either consider it to be a positive angle having a number of degrees equal to 180° + A′OB, or a negative angle having a number of degrees equal to 180° — A′OB. These are, in fact, only two values, which differ by 360° (*v.* § 18). The angle AOB″ also (Figs. 10–13) can be considered equal to + 270° (180° + A′OB″), or equal to — 90° [—(180° — A′OB″)].

In Fig. 10, where OC is supposed to be perpendicular to OB, we have the angle AOC = φ — 90°. This equation indicates that if the rotating line rotates in the positive direction by an amount equal to φ (i.e., to OB), and then in the negative direction by an amount equal to 90°, it will finally take the position OC, and will make with the initial line (OA) the angle AOC, which will be positive or negative, according as φ is greater or less than 90°.

§ **18.** If *to* the angle φ (Figs. 10–13), we add 360° once, twice, or any integral number of times, the resulting angle will still be bounded by the lines OA and OB. If *from* the angle φ we subtract 360° once, twice, or any integral number of times, the resulting angle will still be bounded by the lines OA and OB.

The expression AOB, then, may represent any one of a series of angles; one angle of this series is the angle φ, and all the other angles of the series differ from it by some multiple of 360°. Thus $\text{AOB} = \varphi + k\,360°$, where k stands for 0, or any integer positive or negative. This principle is often important in the addition of angles.

§ **19.** Bearing in mind what has been said about the addition of angles (§ 16), and also the contents of the preceding section, we see at once the truth and significance of the following equations obtained from figures 14 and 15.

FIG. 14.

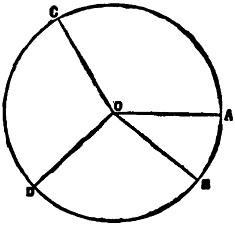
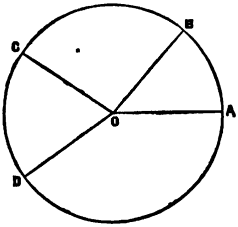


FIG. 15



$$\begin{aligned}
&\mathrm{AOD} + \mathrm{DOA} = k\,360^\circ,\\
&\mathrm{ACD} + \mathrm{DOC} + \mathrm{COB} + \mathrm{BOA} = k\,360^\circ,\\
&\mathrm{AOB} + \mathrm{BOD} + \mathrm{DOC} + \mathrm{COA} = k\,360^\circ,\\
&\mathrm{AOC} + \mathrm{COB} + \mathrm{BOD} + \mathrm{DOA} = k\,360^\circ.
\end{aligned}$$

where k, as in § 18, denotes any integer, positive, negative, or null.

In like manner, if α, β, γ, δ, are any lines in one plane,—

$${}^{\beta}_{\alpha} + {}^{\gamma}_{\beta} + {}^{\delta}_{\gamma} + {}^{\alpha}_{\delta} = k\,360^\circ$$

§ **20.** Before proceeding further we must lay down rules concerning the directions of certain lines which we shall have to treat of in their relations to angles.

a. When speaking of an angle, we shall always regard the vertex as a point of reference or origin with respect to this angle; and we will define the directions of the initial and terminal lines (§ 12) to be always positive, when considered relatively to the angle. Thus, when we speak of the angle between two lines, we always mean the angle between their positive directions.

FIG. 16.

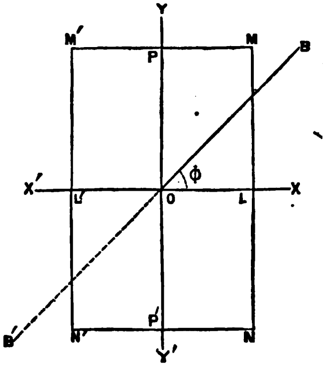
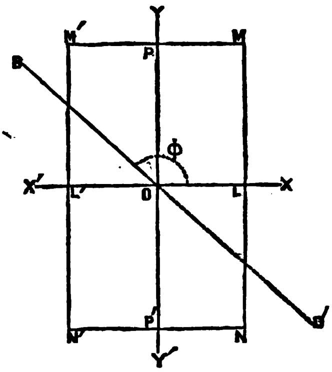


FIG. 17.



In each of the figures 16, 17, 18, and 19, for example, O, the vertex of the angle φ, will be an origin, and the directions of the initial and terminal lines OX and OB will be positive.

FIG. 18.

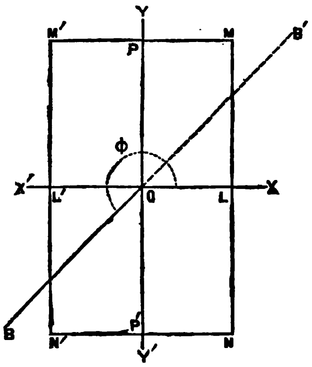
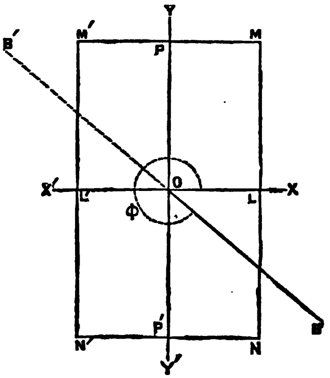


FIG. 19.

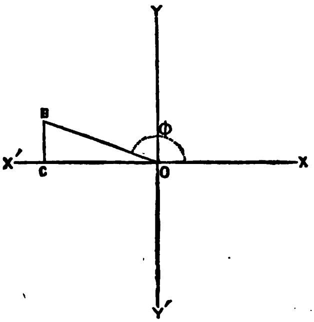


b. Through O, making with the initial line OX an angle of + 90°, draw OY. The lines XX' and YY' are called *axes*. We define OX and OY as the positive directions of the axes; OX' and OY' are therefore their negative directions. These are also the positive and negative directions of all lines parallel to the axes, when considered relatively to them. Thus PM and P'N are positive, while PM' and P'N' are negative; and LM and L'M' are positive, while LN and L'N' are negative. Relatively to the angle XOB, OB is positive, and OB' negative; but relatively to the angle XOB', OB is negative, and OB' positive.

c. Whenever in the same problem we have to treat of two or more angles whose vertices and initial lines do not coincide, we can either refer all the angles to the same origin and to the same initial line, or we can regard each vertex as an origin, and each initial and terminal line as positive, ***with respect to the angle to which it belongs.*** If the latter method is adopted, care must be taken not to confound a direction which is positive with respect to one angle with a direction which is positive with respect to a different angle; for these directions *may* be, when compared with one another, directly opposite.

d. When we have $\varphi = 180°$, or $\varphi = 270°$, the terminal line (OB) will have the negative direction of the axis: we shall, however, still consider OB to be positive, when regarded as the terminal line of φ. Suppose, for example, we have the right triangle BOC (Fig. 18). The base OC is negative; and the hypothenuse OB and perpendicular CB are positive. Now, suppose φ to increase until OB coincides with OC. We shall now regard our right triangle to be one whose perpendicular is zero, whose base OC is negative, and whose hypothenuse OB is, as before, positive.

FIG. 20.



§ **21.** The following principles of Plane Geometry must be borne in mind: —

a. The sum of the angles of a plane triangle is equal to 180°.

b. The sum of the two acute angles of a plane right triangle is equal to 90°. Either of these angles is then the complement (§ 5) of the other.

c. When two plane right triangles have an acute angle and a side of the one equal to an acute angle and the homologous side of the other, the triangles are equal, and all of their homologous parts are equal.

d. When two plane right triangles have an acute angle of the one equal to an acute angle of the other, the triangles are similar, and their homologous sides are therefore proportional.

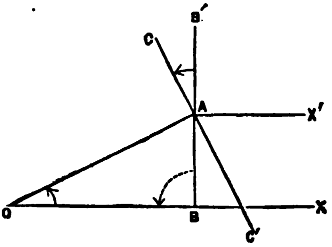
e. The square on the hypothenuse of a right triangle is equal to the sum of the squares on the other two sides.

f. In any triangle the greater side is opposite the greater angle, and the converse.

g. The sum of two sides of a triangle is greater than the third side.

h. Two angles are equal when their sides are respectively perpendicular; but we must be careful to take the sides of the respective angles in the same order, and to measure the angles in the same direction. (*v.* § 14.) In Fig. 21, for example, the angle B′AC = angle XOA, for both angles are measured in the positive (§ 14) direction, and B′AC is formed by AB′, which is perpendicular to OX, and AC, which is perpendicular to OA. Similarly, angle BAC′ = XOA, and angle X′AB′ = ABO.

FIG. 21.



i. The chord of an arc of 60° is equal to the radius.

EXAMPLES.

1. To what quadrant does each of the following angles belong? 289°, 368°, 510°, 640°, 1178°, — 80°, — 188°, — 1722°.

2. Represent by figures the following angles, where in each case $\varphi = 60^\circ$, and $\theta = 30^\circ$; $\varphi - 360^\circ$, $\varphi - 90^\circ$, $\varphi - \theta + 180^\circ$, $\varphi - 270^\circ$, $\varphi + \theta + 60^\circ$.

3. Change to circular measure each of the following angles: 88° 2′, 271° 53′ — 18° 7′, 60°, 390°.

4. The circular measures of certain angles are as follows: 2., 16.2, 6., $-\frac{\pi}{8}$; find in each case the corresponding number of degrees.

5. Find the lengths of the following arcs in a circle whose radius is unity: 60°, 30°, 150°, 330°, 18°, 268°, 135°.

6. Find the lengths of the above arcs in a circle whose radius is 6.

7. Taking the earth's equatorial radius to be 3963 miles, find the length of an arc of 1° on the equator.

CHAPTER II.

THE TRIGONOMETRIC FUNCTIONS.

§ **22.** A right triangle can always be formed by letting fall a perpendicular from *any* point in the terminal line of a given angle upon the initial line, produced if necessary. This triangle we will call a triangle of reference for the angle; and, in applying the rules of § 20 to determine the signs of its sides, we shall always regard the hypothenuse and base as measured *from* the vertex of the given angle, and the perpendicular as measured *from* the initial line.

In each of the figures 22, 23, 24, and 25, for example,

FIG. 22.

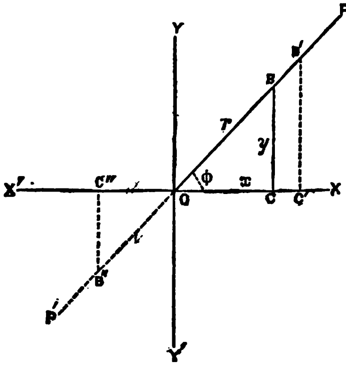


FIG. 23.

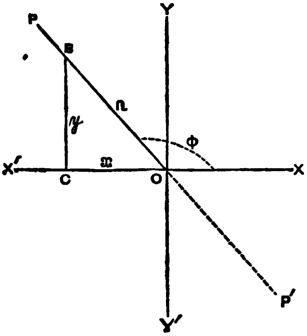


FIG. 24.

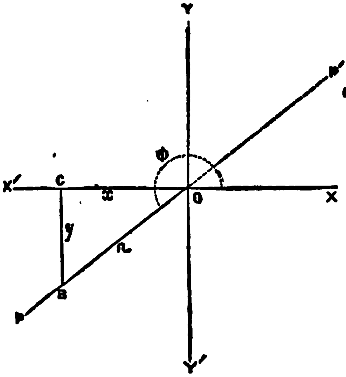
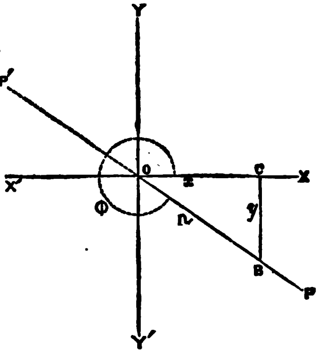


FIG. 25.



where φ (XOP) is, in each case, the given angle, we can form the right triangle OCB, as stated above. In order to form this triangle in figures 23 and 24, we must produce the line OX in the direction OX′.

In the above figures, the letters x, y, and r represent the base (OC), the perpendicular (CB), and the hypothenuse (OB) respectively, both in direction and length. Therefore x is positive in figures 22 and 25, and negative in figures 23 and 24; y is positive in figures 22 and 23, and negative in figures 24 and 25; r is positive in each figure. But r might be made negative in each figure. Thus, in Fig. 22, we may take $x = \text{OC}''$, $y = \text{C}''\text{B}''$, $r = \text{OB}''$, making x, y, and r all negative.

§ **23.** The *Trigonometric Functions* of an angle are the six ratios between the three sides of a triangle formed as above, taking into account the signs as well as the lengths of these sides.

These ratios are always the same for the same angle, both in numerical value and in sign; for if, as in figure 22, we let fall perpendiculars from different points of the terminal line, as B, B′, and B″, we have the right triangles OCB, OC′B′, and OC″B″, which evidently have their acute angles at O equal. These triangles are therefore (*v.* § 21, *d*) similar, and their homologous sides are proportional; i.e., the six ratios of the sides of OCB are, in numerical value, respectively equal to the six ratios of the sides of OC′B′, or OC″B″. They are equal in sign as well; for, in comparing any two of the triangles, the sides of the first either have the same directions, and therefore the same signs, as the corresponding sides of the second, as in OCB and OC′B′; or, as in OCB and OC″B″, each side of the second has a direction, and therefore a sign, opposite to that of the corresponding side of the first; and in each case the signs of the *ratios* are obviously the same.

These ratios are, in general, different for different angles; for two right triangles, *not* having an acute angle of the one *equal* to an acute angle of the other, are *not similar*, and their homologous sides are *not proportional.*

Thus we see that the values of the trigonometric ratios of an angle *depend*, and depend *only*, upon the value of the angle; for this reason they are called the *trigonometric functions** of the angle.

As soon as we have given names to these ratios, we shall be able to introduce into the same formula two entirely different kinds of quantities,—angles and straight lines. We shall then have made a very important step in the treatment of our subject.

---

* When one quantity depends upon another for its value, so that a change in the second necessitates a change in the first, the first is called a *function* of the second.

§ **24.** To each of the six ratios mentioned above, a name has been given as follows: In each of the figures 22, 23, 24, and 25, —

			is called the	abbreviated into
$\frac{\text{CB}}{\text{OB}}$	$\frac{y}{r}$	$\frac{\text{perpendicular}}{\text{hypothenuse}}$	sine of φ	$\sin\varphi$
$\frac{\text{OC}}{\text{OB}}$	$\frac{x}{r}$	$\frac{\text{base}}{\text{hypothenuse}}$	cosine of φ	$\cos\varphi$
$\frac{\text{CB}}{\text{OC}}$	$\frac{y}{x}$	$\frac{\text{perpendicular}}{\text{base}}$	tangent of φ	$\tan\varphi$
$\frac{\text{OC}}{\text{CB}}$	$\frac{x}{y}$	$\frac{\text{base}}{\text{perpendicular}}$	cotangent of φ	$\text{ctn}\,\varphi$
$\frac{\text{OB}}{\text{OC}}$	$\frac{r}{x}$	$\frac{\text{hypothenuse}}{\text{base}}$	secant of φ	$\sec\varphi$
$\frac{\text{OB}}{\text{CB}}$	$\frac{r}{y}$	$\frac{\text{hypothenuse}}{\text{perpendicular}}$	cosecant of φ	$\csc\varphi$

In addition to the above

$1 - \cos\varphi$ versed sine of φ $\text{vers}\,\varphi$

§ **25.** Bearing in mind what has been said about the signs of x, y, and r, in § 22, we easily deduce the following table for the signs of the functions of angles in each quadrant: —

φ	sin	cos	tan	ctn	sec	csc
1st quad	+	+	+	+	+	+
2d quad	+	—	—	—	—	+
3d quad	—	—	+	+	—	—
4th quad	—	+	—	—	+	—

Since the hypothenuse of a right triangle is the greatest of the three sides, the first two of our six ratios never exceed 1 in numerical value, and the last two never fall below 1 in numerical value. Hence *the sine and cosine vary only from* -1 *to* $+1$; *the secant and cosecant vary only from* $+1$ *to* $+\infty$ *and from* -1 *to* $-\infty$; *while the tangent and cotangent may have any value from* $-\infty$ *to* $+\infty$.

§ **26.** By § 24 we have, —

$$\left.\begin{array}{lcl}
\sin\varphi=\dfrac{y}{r}, & & \sin\varphi=\dfrac{1}{\csc\varphi}, \\
 & \therefore \sin\varphi\times\csc\varphi=1 \therefore & \\
\csc\varphi=\dfrac{r}{y}, & & \csc\varphi=\dfrac{1}{\sin\varphi}, \\
\cos\varphi=\dfrac{x}{r}, & & \cos\varphi=\dfrac{1}{\sec\varphi}, \\
 & \therefore \cos\varphi\times\sec\varphi=1 \therefore & \\
\sec\varphi=\dfrac{r}{x}, & & \sec\varphi=\dfrac{1}{\cos\varphi}, \\
\tan\varphi=\dfrac{y}{x}, & & \tan\varphi=\dfrac{1}{\operatorname{ctn}\varphi}, \\
 & \therefore \tan\varphi\times\operatorname{ctn}\varphi=1 \therefore & \\
\operatorname{ctn}\varphi=\dfrac{x}{y}, & & \operatorname{ctn}\varphi=\dfrac{1}{\tan\varphi},
\end{array}\right\} \quad [1]$$

Therefore the *sine*, *cosine*, and *tangent* are reciprocals of the *cosecant*, *secant*, and *cotangent* respectively.

§ **27.** $$\frac{\sin\varphi}{\cos\varphi}=\frac{y}{r}\div\frac{x}{r}=\frac{y}{x}=\tan\varphi,\ \therefore\frac{\sin\varphi}{\cos\varphi}=\tan\varphi. \quad [2]$$

$$\frac{\cos\varphi}{\sin\varphi}=\frac{x}{r}\div\frac{y}{r}=\frac{x}{y}=\operatorname{ctn}\varphi,\ \therefore\frac{\cos\varphi}{\sin\varphi}=\operatorname{ctn}\varphi. \quad [3]$$

Any two of the quantities appearing in either [2] or [3] being given, it is evident that the third can be found.

§ **28.** *Relations between sine and cosine, tangent and secant, cotangent and cosecant.*

By § 21 *e*, we have (Figs. 22–25) $x^2+y^2=r^2$, a formula which holds for all the figures since the *squares* of x, y, and r are positive in all cases; and from this we get: —

$$\frac{y^2}{r^2}+\frac{x^2}{r^2}=1, \therefore \sin^2\varphi+\cos^2\varphi=1, \qquad [4]$$

$$\frac{r^2}{x^2}=1+\frac{y^2}{x^2}, \therefore \sec^2\varphi=1+\tan^2\varphi, \qquad [5]$$

$$\frac{r^2}{y^2}=1+\frac{x^2}{y^2}, \therefore \csc^2\varphi=1+\operatorname{ctn}^2\varphi, \qquad [6]$$

The above formulas enable us to solve the following problems: —

To find $\cos\varphi$ when $\sin\varphi$ is given, and the converse.

$\sec\varphi$ " $\tan\varphi$ " " "

$\csc\varphi$ " $\operatorname{ctn}\varphi$ " " "

In order, however, to solve these problems completely, it is also necessary to know to what quadrant φ belongs. When, for example, $\cos\varphi$ is given, we have from [4] $\sin\varphi=\pm\sqrt{1-\cos^2\varphi}$, which does not, on account of the double sign of the radical, completely determine $\sin\varphi$; but, when we also know to what quadrant φ belongs, we know by § 25 the sign of its sine, and can therefore completely determine $\sin\varphi$. It should be noted that formulas [1]–[6] enable us, when any one function is given, to find all the others. The student is advised, before reading farther, to do examples 1–12 at the end of this chapter.

§ **29.** *To represent the trigonometric functions by straight lines.*

In order that the trigonometric functions, or ratios, may be represented by straight lines, the triangle of reference must be so taken for each ratio that the denominator of the ratio shall be unity.

In Figs. 26–29, let φ, or the angle XOP, be an angle of the 1st, 2d, 3d, and 4th quadrants respectively. From O as a centre, with a radius unity, describe a circle.

FIG. 26.

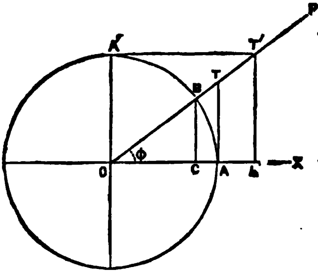


FIG. 27.

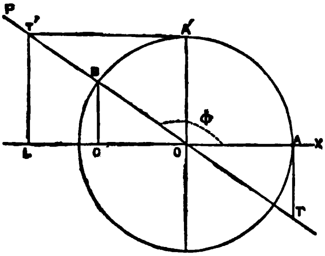


FIG. 28.

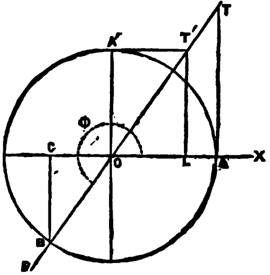
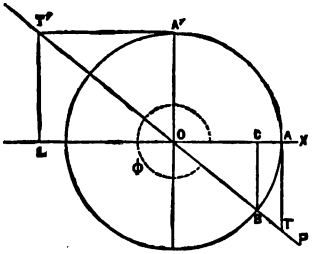


FIG. 29.



$\sin\varphi = \frac{\text{perp}}{\text{hyp}}$, $\cos\varphi = \frac{\text{base}}{\text{hyp}}$. Since the hypothenuse must be unity, we will, in each figure, construct the triangle of reference OCB, in which the hypothenuse OB is equal to the radius, or unity.

$$\therefore \sin\varphi = \frac{CB}{1} = CB, \cos\varphi = \frac{OC}{1} = OC.$$

$\tan\varphi = \frac{\text{perp}}{\text{base}}$, $\sec\varphi = \frac{\text{hyp}}{\text{base}}$. By drawing a geometric tangent at A, we have the triangle of reference OAT, in which the base OA is unity.

$$\therefore \tan\varphi = \frac{AT}{1} = AT, \sec\varphi = \frac{OT}{1} = OT.$$

$\text{ctn}\,\varphi = \frac{\text{base}}{\text{perp}}$, $\csc\varphi = \frac{\text{hyp}}{\text{perp}}$. Through a point $+ 90°$ from A, draw a geometric tangent which will cut OB produced in some point as T′; from T′ let fall a perpendicular upon X′X. The triangle of reference OLT′ thus formed will be a triangle in which the perpendicular $LT' = OA' = 1$, and the base $OL = A'T'$.

$$\therefore \text{ctn}\,\varphi = \frac{OL}{1} = OL = A'T', \csc\varphi = \frac{OT'}{1} = OT'.$$

$$\text{versed}\sin\varphi = 1 - \cos\varphi = OA - OC = CA$$

From § 20 and § 22 we know that: —

In Fig. 26, —

OC, CB, AT, OT, A′T′, OT′, and CA are positive.

In Fig. 27, —

CB, OT′, and CA are positive,
and OC, AT, OT, and A′T′ are negative.

In Fig. 28, —

AT, A′T′, and CA are positive,
and CB, OC, OT, and OT′ are negative.

In Fig. 29, —

OC, OT, and CA are positive,
and CB, AT, A′T′, and O′T′ are negative.

Putting the above results in words, we have the following corollaries to our original definitions (*v.* § 24): —

In a circle whose radius is unity, the *trigonometric functions* of an angle or arc may be *represented* by the following lines: —

The *sine*, by the line which measures the perpendicular distance of the end of the arc *from*[*] the initial line.

The *cosine*, by the line drawn *from* the centre of the circle *to* the foot of the *sine*.

The *tangent*, by that part of the geometric tangent at the beginning of the arc which is drawn *from* the point of tangency *to* the terminal line.

The *cotangent*, by that part of the geometric tangent at a point +90° from the beginning of the arc which is drawn *from* the point of tangency *to* the terminal line.

The *secant*, by the line drawn *from* the centre of the circle *to* the end of the tangent.

The *cosecant*, by the line drawn *from* the centre of the circle *to* the end of the cotangent.

The *versed sine*, by the line drawn *from* the foot of the sine to the beginning of the arc.

What has been said in this section should not lead the student to regard the trigonometric functions as absolute lines. The trigonometric functions are *not* the lines themselves, but are merely the *numbers* which denote the ratios of the lines to the arbitrary unit which we choose as a radius.

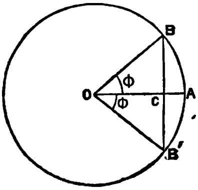
---

[*] The word "from" indicates the direction, and therefore the sign of the line.

§ **30.** *In any circle the radius of which is unity, the sine of any arc is one-half the chord of twice the arc.*

In Fig. 30 let the angle B′OB or the arc $B'B = 2\varphi$. Draw OA so as to bisect the angle or 2φ; draw the chord B′B. Since OA bisects the angle B′OB, we know from geometry that it is perpendicular to and bisects the chord B′B. ∴ $CB = \frac{1}{2} B'B = \frac{1}{2}$ chord 2φ.

FIG. 30.

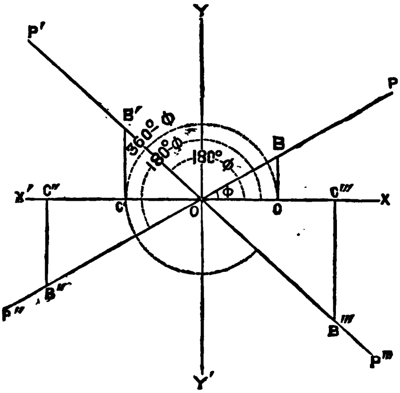


But by § 29, —

$$CB = \sin\varphi \therefore \sin\varphi = \tfrac{1}{2} \text{ chord } 2\varphi \qquad [7]$$

§ **31.*** *Relations between the functions of $\varphi \pm k\,360°$, $180° - \varphi$, $180° + \varphi$, $360° - \varphi$, $-\varphi$, and the functions of φ.*

FIG. 31.



Let us take any angle of the 1st quadrant, as XOP (Fig. 31); let the acute value of this angle be denoted by φ. XOP then denotes either the angle φ or the angle $\varphi \pm k\,360°$ (*v.* § 18).

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* A treatment of the subject of §§ 31, 32, and 33, covering explicitly all possible values of φ, is given in appendix I., which the teacher may at his discretion substitute for these sections.

Construct the angles:—

$$\mathrm{XOP}' = 180^\circ - \varphi, \quad \mathrm{XOP}'' = 180^\circ + \varphi,$$
$$\mathrm{XOP}''' = 360^\circ - \varphi \text{ or } = -\varphi.$$

In accordance with this construction, the acute angles XOP or φ, $\mathrm{P'OX'}$, $\mathrm{X'OP''}$, and $\mathrm{P'''OX}$ are all equal to one another.

Construct the triangles of reference OCB, $\mathrm{OC'B'}$, $\mathrm{OC''B''}$, and $\mathrm{OC'''B'''}$; each of these triangles having an acute angle equal to φ, they are similar (*v.* § 21, *d*), and their homologous sides are proportional; moreover, the perpendiculars CB, $\mathrm{C'B'}$, $\mathrm{C''B''}$, and $\mathrm{C'''B'''}$, being opposite the equal angles, are homologous. Therefore the corresponding trigonometric functions of the angles φ, $\varphi \pm k\, 360^\circ$, $180^\circ - \varphi$, $180^\circ + \varphi$, $360^\circ - \varphi$, and $-\varphi$, are numerically equal; but, paying proper attention to the algebraic signs of these functions (*v.* § 25), we arrive at the following results:—

$$\left.\begin{array}{l} \sin(\varphi \pm k\, 360^\circ) = \sin\varphi \\ \cos(\varphi \pm k\, 360^\circ) = \cos\varphi \end{array}\right\} \qquad [8]$$

$$\left.\begin{array}{lll} \dfrac{\mathrm{C'B'}}{\mathrm{OB'}} = & \dfrac{\mathrm{CB}}{\mathrm{OB}} & \therefore \sin(180^\circ - \varphi) = \sin\varphi \\ \dfrac{\mathrm{OC'}}{\mathrm{OB'}} = & -\dfrac{\mathrm{OC}}{\mathrm{OB}} & \therefore \cos(180^\circ - \varphi) = -\cos\varphi \end{array}\right\} \qquad [9]$$

$$\left.\begin{array}{lll} \dfrac{\mathrm{C''B''}}{\mathrm{OB''}} = & -\dfrac{\mathrm{CB}}{\mathrm{OB}} & \therefore \sin(180^\circ + \varphi) = -\sin\varphi \\ \dfrac{\mathrm{OC'}}{\mathrm{OB''}} = & -\dfrac{\mathrm{OC}}{\mathrm{OB}} & \therefore \cos(180^\circ + \varphi) = -\cos\varphi \end{array}\right\} \qquad [10]$$

$$\left.\begin{array}{lll} \dfrac{\mathrm{C'''B'''}}{\mathrm{OB'''}} = & -\dfrac{\mathrm{CB}}{\mathrm{OB}} & \therefore \sin(360^\circ - \varphi) = -\sin\varphi \\ & & \qquad\qquad\qquad\quad = \sin(-\varphi) \\ \dfrac{\mathrm{OC'''}}{\mathrm{OB'''}} = & \dfrac{\mathrm{OC}}{\mathrm{OB}} & \therefore \cos(360^\circ - \varphi) = \cos\varphi \\ & & \qquad\qquad\qquad\quad = \cos(-\varphi) \end{array}\right\} \qquad [11]$$

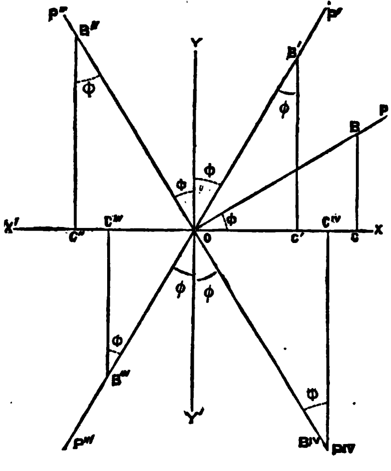
In like manner the other functions of these angles may be found; they may also be found from the above by [1], [2], and [3].

§ **32.** Relations between the functions of $90^\circ + \varphi$, $90^\circ - \varphi$, $270^\circ + \varphi$, and $270^\circ - \varphi$.

Let us take any angle of the 1st quadrant, as XOP (Fig. 32), which we will call φ. Construct the angles,—

$$XOP' = 90^\circ - \varphi,\quad XOP'' = 90^\circ + \varphi,$$
$$XOP''' = 270^\circ - \varphi,\quad XOP^{iv} = 270^\circ + \varphi.$$

FIG. 32.



Construct, with respect to these angles, the triangles of reference OCB, $OC'B'$, $OC''B''$, $OC'''B'''$, and $OC^{iv}B^{iv}$. In accordance with this construction, the angles marked φ in the figure are all equal.

The triangles OCB, $OC'B'$, $OC''B''$, $OC'''B'''$, and $OC^{iv}B^{iv}$, having each an acute angle equal to φ, are all similar (*v.* § 21, *d*), and their homologous sides are proportional. But in this case the lines OC', OC'', OC''', and OC^{iv} of the triangles $OC'B'$, $OC''B''$, $OC'''B'''$, and $OC^{iv}B^{iv}$, being opposite the angle φ, are homologous with CB of the triangle OCB; and $C'B'$, $C''B''$, $C'''B'''$, and $C^{iv}B^{iv}$, are homologous with OC. Therefore, paying proper attention to the algebraic signs, we get the following results:—

$$\left.\begin{array}{lll} \frac{C'B'}{OB'} = \frac{OC}{OB} & \therefore \sin(90° - \varphi) = & \cos\varphi \\ \frac{OC'}{OB'} = \frac{CB}{OB} & \therefore \cos(90° - \varphi) = & \sin\varphi \end{array}\right\} \qquad [12]$$

$$\left.\begin{array}{lll} \frac{C''B''}{OB''} = \frac{OC}{OB} & \therefore \sin(90° + \varphi) = & \cos\varphi \\ \frac{OC''}{OB''} = -\frac{CB}{OB} & \therefore \cos(90° + \varphi) = & -\sin\varphi \end{array}\right\} \qquad [13]$$

$$\left.\begin{array}{lll} \frac{C'''B'''}{OB'''} = -\frac{OC}{OB} & \therefore \sin(270° - \varphi) = & -\cos\varphi \\ \frac{OC'''}{OB'''} = -\frac{CB}{OB} & \therefore \cos(270° - \varphi) = & -\sin\varphi \end{array}\right\} \qquad [14]$$

$$\left.\begin{array}{lll} \frac{C^{iv}B^{iv}}{OB^{iv}} = -\frac{OC}{OB} & \therefore \sin(270° + \varphi) = & -\cos\varphi \\ \frac{OC^{iv}}{OB^{iv}} = \frac{CB}{OB} & \therefore \cos(270° + \varphi) = & \sin\varphi \end{array}\right\} \qquad [15]$$

§ **33.** From [12], by [1], [2], and [3], we readily obtain the following:—

$$\left.\begin{array}{l} \csc(90° - \varphi) = \sec\varphi \\ \sec(90° - \varphi) = \csc\varphi \\ \tan(90° - \varphi) = \text{ctn}\,\varphi \\ \text{ctn}\,(90° - \varphi) = \tan\varphi \end{array}\right\} \qquad [12_1]$$

Now, φ and $90° - \varphi$ are complements of each other: therefore from [12] and [12_1] we see that the *sine*, *tangent*, and *secant* of any angle are respectively the *cosine*, *cotangent*, and *cosecant* of its complement.

"Co", then, is merely an abbreviation for the word "complement's;" [12] and [12_1] may therefore be expressed in words as follows:—

The *cosine* of any angle is its *complement's sine.*

The *cosecant* of any angle is its *complement's secant.*

The *cotangent* of any angle is its *complement's tangent.*

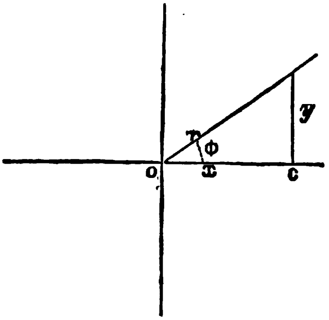
Formulas 8–15 are easily remembered if we note that when φ is coupled with 0°, 180°, or k 360°, the functions of the angles thus formed, and those of φ, are *numerically the same;* while, when φ is coupled with 90° or 270°, the functions of the angles thus formed, and those of φ, are *numerically complementary.*

The *algebraic sign* is determined in each case by supposing φ to be acute, and then noticing to what quadrant the resulting angle belongs.

It will be readily seen, by constructing proper figures, that formulas 8–15 hold for all values of φ.

§ **34.** *To find the functions of 0° and 360°.*

FIG. 33.



Suppose φ (Fig. 33) to decrease until it becomes 0; then y will become 0, and r will coincide with x.

$$\therefore \sin 0° = \frac{y}{r} = \frac{0}{r} = 0,$$

$$\cos 0° = \frac{x}{r} = 1,$$

$$\tan 0° = \frac{y}{x} = \frac{0}{x} = 0, \qquad \sec 0° = \frac{r}{x} = 1,$$

$$\text{ctn } 0° = \frac{x}{y} = \frac{x}{0} = \infty, \qquad \csc 0° = \frac{r}{y} = \frac{r}{0} = \infty.$$

When φ is equal to 360° we shall also have $y = 0$, and $r = x$; therefore the functions of 360° are the same as those of 0°.

§ 35. *To find the functions of 90°.*

Suppose (Fig. 33) φ to increase until it becomes equal to 90°; then x will become 0, and y will coincide with r.

$$\therefore \sin 90^\circ = \frac{y}{r} = 1, \qquad \csc 90^\circ = \frac{r}{y} = 1,$$

$$\cos 90^\circ = \frac{x}{r} = \frac{0}{r} = 0, \qquad \sec 90^\circ = \frac{r}{x} = \frac{r}{0} = \infty,$$

$$\tan 90^\circ = \frac{y}{x} = \frac{y}{0} = \infty \qquad \text{ctn}\ 90^\circ = \frac{x}{y} = \frac{0}{y} = 0.$$

These results can also be found by putting $\varphi = 0$ in [12] and [12_1].

§ 36. *To find the functions of 180°.*

Suppose φ (Fig. 33) to increase until it becomes 180°: then y will become 0, and r will become *numerically* equal to x, but opposite in *sign*. (*v.* § 20, *d.*)

$$\therefore \sin 180^\circ = \frac{y}{r} = \frac{0}{r} = 0, \quad \csc 180^\circ = \frac{r}{y} = \frac{r}{0} = \infty,$$

$$\cos 180^\circ = \frac{x}{r} = -1, \quad \tan 180^\circ = \frac{y}{x} = \frac{0}{x} = 0,$$

$$\sec 180^\circ = \frac{r}{x} = -1, \quad \text{ctn}\ 180^\circ = \frac{x}{y} = \frac{x}{0} = \infty.$$

Sin 180° and cos 180° may also be found by putting $\varphi = 0$ in [9], and from these the other functions by [1], [2], and [3].

§ **37.** *To find the functions of 270°.*

Suppose φ (Fig. 33) to increase until it becomes 270°: then x will become 0, and r will become *numerically* equal to y, but opposite in *sign*. (*v.* § 20, *d.*)

$$\therefore \sin 270° = \frac{y}{r} = -1 \qquad \csc 270° = \frac{r}{-y} = -1$$

$$\cos 270° = \frac{x}{r} = \frac{0}{r} = 0 \qquad \sec 270° = \frac{r}{0} = \infty$$

$$\tan 270° = \frac{y}{x} = \frac{y}{0} = \infty \qquad \text{ctn}\, 270° = \frac{x}{y} = \frac{0}{y} = 0$$

§ **38.** *To find the functions of 30° and 60°.*

In formula 7 let $\varphi = 30°$; this formula will then become $\sin 30° = \frac{1}{2}$ chord 60°. But by § 21, *i*, chord 60° = radius = 1. $\therefore \sin 30° = \frac{1}{2}$. $\therefore$ from [4] we have, —

$$\cos 30° = +\sqrt{1 - \sin^2 30} = +\sqrt{1 - \tfrac{1}{4}} = +\sqrt{\tfrac{3}{4}}$$
$$= +\tfrac{1}{2}\sqrt{3};\text{*}$$

and from [2], —

$$\tan 30° = \frac{\sin 30}{\cos 30} = \frac{\frac{1}{2}}{\frac{1}{2}\sqrt{3}} = \frac{1}{\sqrt{3}};$$

from [3], —

$$\text{ctn}\, 30° = \frac{1}{\tan 30°} = \sqrt{3};$$

from [1], —

$$\sec 30° = \frac{1}{\cos 30} = \frac{2}{\sqrt{3}}, \ \csc 30° = \frac{1}{\sin 30} = 2.$$

---

* Why +? *v.* remark at end of § 28.

We have then by [12] and [12_1] since

$$60° = 90° - 30°;$$

$$\sin 60 = \cos 30° = \tfrac{1}{2}\sqrt{3}, \quad \cos 60° = \sin 30° = \tfrac{1}{2},$$

$$\tan 60° = \operatorname{ctn} 30° = \sqrt{3}, \quad \operatorname{ctn} 60° = \tan 30° = \frac{1}{\sqrt{3}},$$

$$\sec 60° = \csc 30° = 2, \quad \csc 60° = \sec 30° = \frac{2}{\sqrt{3}}.$$

§ 39. *To find the functions of 45°.*

45° is the complement of 45°. Therefore by [12] and [12_1], —

$$\sin 45 = \cos 45, \quad \tan 45° = \operatorname{ctn} 45°, \quad \sec 45° = \csc 45°.$$

Whence by [4], —

$$\sin^2 45° + \cos^2 45° = 1 = 2\sin^2 45° = 2\cos^2 45°.$$

$$\therefore \sin 45° = \cos 45° = +\sqrt{\frac{1}{2}},^*$$

$$\tan 45° = \operatorname{ctn} 45° = \frac{\sin 45°}{\cos 45°} = 1,$$

$$\sec 45° = \csc 45° = \frac{1}{\sin 45°} = \sqrt{2}.$$

§ 40. *To find the functions of 120°.*

$$120° = 180° - 60°;$$

$\therefore$ by [8], —

$$\sin 120° = \sin 60° = \tfrac{1}{2}\sqrt{3},$$

$$\cos 120° = -\cos 60° = -\tfrac{1}{2},$$

---

* Why +? (v. remark at end of § 28.)

$$\tan 120^\circ = \frac{\sin 120^\circ}{\cos 120^\circ} = \frac{\frac{1}{2}\sqrt{3}}{-\frac{1}{2}} = -\sqrt{3},$$

$$\operatorname{ctn} 120^\circ = \frac{1}{\tan 120^\circ} = -\frac{1}{\sqrt{3}},$$

$$\sec 120^\circ = \frac{1}{\cos 120^\circ} = \frac{1}{-\frac{1}{2}} = -2,$$

$$\csc 120^\circ = \frac{1}{\sin 120^\circ} = \frac{1}{\frac{1}{2}\sqrt{3}} = \frac{2}{\sqrt{3}}.$$

§ 41. *To find the functions of 150°.*

$$150^\circ = 180^\circ - 30^\circ$$

∴ by [8], —

$$\sin 150^\circ = \sin 30^\circ = \tfrac{1}{2}$$

$$\cos 150^\circ = -\cos 30^\circ = -\tfrac{1}{2}\sqrt{3},$$

$$\tan 150^\circ = \frac{\sin 150^\circ}{\cos 150^\circ} = \frac{\frac{1}{2}}{-\frac{1}{2}\sqrt{3}} = -\frac{1}{\sqrt{3}},$$

$$\operatorname{ctn} 150^\circ = \frac{1}{\tan 150^\circ} = -\sqrt{3},$$

$$\sec 150^\circ = \frac{1}{\cos 150^\circ} = -\frac{2}{\sqrt{3}},$$

$$\csc 150^\circ = \frac{1}{\sin 150^\circ} = \frac{1}{\frac{1}{2}} = 2.$$

§ 42. *To find the functions of 135°.*

$$135^\circ = 180^\circ - 45^\circ;$$

∴ by [8], —

$$\sin 135^\circ = \sin 45^\circ = \sqrt{\tfrac{1}{2}}\,,$$

$$\cos 135^\circ = -\cos 45^\circ = -\sqrt{\tfrac{1}{2}},$$

$$\tan 135^\circ = \frac{\sin 135^\circ}{\cos 135^\circ} = \frac{\sqrt{\tfrac{1}{2}}}{-\sqrt{\tfrac{1}{2}}} = -1,$$

$$\operatorname{ctn} 135^\circ = \frac{1}{\tan 135^\circ} = \frac{1}{-1} = -1,$$

$$\sec 135^\circ = \frac{1}{\cos 135^\circ} = \frac{1}{-\sqrt{\tfrac{1}{2}}} = -\sqrt{2},$$

$$\csc 135^\circ = \frac{1}{\sin 135^\circ} = \frac{1}{\sqrt{\tfrac{1}{2}}} = \sqrt{2}.$$

§ **43.** Collecting the results of the last nine sections for easy reference, we have: —

Angle	sin	cos	tan	ctn	sec	csc
0°	0	1	0	∞	1	∞
30°	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$\sqrt{\frac{1}{3}}$	$\sqrt{3}$	$2\sqrt{\frac{1}{3}}$	2
45°	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$\sqrt{3}$	$\sqrt{\frac{1}{3}}$	2	$2\sqrt{\frac{1}{3}}$
90°	1	0	∞	0	∞	1
120°	$\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$	$-\sqrt{3}$	$-\sqrt{\frac{1}{3}}$	-2	$2\sqrt{\frac{1}{3}}$
135°	$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$	-1	-1	$-\sqrt{2}$	$\sqrt{2}$
150°	$\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	$-\sqrt{\frac{1}{3}}$	$-\sqrt{3}$	$-2\sqrt{\frac{1}{3}}$	2
180°	0	-1	0	∞	-1	∞
270°	-1	0	∞	0	∞	-1
360°	0	1	0	∞	1	∞

[16].

§ 44. *Increase and decrease of the functions.*

We have shown in § 30 that, in the circle of which the radius is unity, —

$$\sin\varphi = \tfrac{1}{2}\,\text{chord}\,2\varphi.$$

Now, we know by geometry that the increase of an arc less than a semi-circumference is accompanied by the increase of its chord. Hence, if $\varphi < 90°$, that is, if $2\varphi < 180°$, $\sin\varphi$ increases with the increase of φ. Since, when $\varphi < 90°$, $\cos\varphi = +\sqrt{1-\sin^2\varphi}$, $\cos\varphi$ decreases with the increase of $\sin\varphi$, and therefore decreases with the increase of φ. Again, $\tan\varphi = \frac{\sin\varphi}{\cos\varphi}$, of which the numerator increases and the denominator decreases with the increase of φ. Either of these changes causes the increase of the fraction. Hence, if $\varphi < 90°$, $\tan\varphi$ increases with the increase of φ. The reciprocal of a quantity decreases with the increase of the quantity, and *vice versa.* Hence we see that *the sine, tangent, and secant of an acute angle increase with the increase of the angle, while the cosine, cotangent, and cosecant decrease.*

Now, if φ is acute, $180°-\varphi$, $180°+\varphi$, and $360°-\varphi$, are general expressions for angles in the second, third, and fourth quadrants respectively; and $180°+\varphi$ increases with the increase of φ, while $180°-\varphi$ and $360°-\varphi$ decrease with the increase of φ. But the functions of these angles are *numerically* equal to those of φ, by § 31. Hence, in the third quadrant, the rule for the increase and decrease of the numerical values of the functions is the same as in the first; but in the second and fourth quadrants the rule is opposite to the first.

The *algebraic* value of a negative quantity (that is, the complete value, taking account of the sign) increases with the decrease of its numerical value, and *vice versa.* Regarding, then, the table of signs in § 25, we obtain the following results:—

	sin.	cos.	tan.	ctn.	sec.	csc.
0°	∓ 0	$+1$	∓ 0	$\mp\infty$	$+1$	$\mp\infty$
1st qu.	inc.	dec.	inc.	dec.	inc.	dec.
90°	$+1$	± 0	$\pm\infty$	± 0	$\pm\infty$	$+1$
2d qu.	dec.	dec.	inc.	dec.	inc.	inc.
180°	± 0	-1	∓ 0	$\mp\infty$	-1	$\pm\infty$
3d qu.	dec.	inc.	inc.	dec.	dec.	inc.
270°	-1	∓ 0	$\pm\infty$	± 0	$\mp\infty$	-1
4th qu.	inc.	inc.	inc.	dec.	dec.	dec.
360°	∓ 0	$+1$	∓ 0	$\mp\infty$	$+1$	$\mp\infty$

[17]

In this table, the sign before ∞ or 0 shows whether the function is passing from $+$ to $-$ or from $-$ to $+$, with the increase of the angle.

§ **45**. *Inverse or anti functions.*

In $x = \sin\varphi$, x is expressed as a function of φ. In order to express φ as a function of x, the following notation has been adopted:—

$$\varphi = \sin^{-1} x.$$

This equation denotes that φ is that angle whose sine is x.

In like manner, if $y = \cos\varphi$, $z = \tan\varphi$, and $w = \sec\varphi$, we can write:—

$$\varphi = \cos^{-1} y, \qquad \varphi = \tan^{-1} z, \qquad \varphi = \sec^{-1} w.$$

$\text{Sin}^{-1} x$, $\cos^{-1} y$, $\tan^{-1} z$, &c., are called the *inverse* or *anti* trigonometric functions, and are read antisine of x, anticosine of y, &c.; they are sometimes written arc sin x, arc cos y, arc tan z, &c., since they denote the *arc*, or angle, whose sine is x, whose cosine is y, &c.

In like manner we can denote the inverse or anti-logarithmic functions. Thus, if $\log u = x$, then $u = \log^{-1} x$.

The inverse of "log sin" may be written, $(\log \sin)^{-1}$. Thus, if $\log \sin \varphi = 9.8421$, then $\varphi = (\log \sin)^{-1} 9.8421$.

Examples: —

$\sin 45° = \sqrt{\tfrac{1}{2}}$, $\tan 30° = \sqrt{\tfrac{1}{3}}$, $\csc 270° = -1$;

therefore,

$45° = \sin^{-1} \sqrt{\tfrac{1}{2}}$, $30° = \tan^{-1} \sqrt{\tfrac{1}{3}}$, $270° = \csc^{-1}(-1)$.

Any relation which can be shown to exist between the trigonometric functions can be expressed by means of the inverse notation. If, for example, —

$$\tan \varphi = \frac{\sin \varphi}{\cos \varphi} = \frac{\sin}{\cos} \varphi = x,$$

$$\text{then} \qquad \varphi = \tan^{-1} x = \left(\frac{\sin}{\cos}\right)^{-1} x.$$

EXAMPLES.

1. What signs belong to the different functions of each of the following angles?

289°, 368°, 510°, 640°, 1178°, —80°, —188°, —1722°.

2. In what quadrants must an angle be taken whose secant $= +\frac{5}{4}$?

In what quadrants must an angle be taken whose tangent $= -\frac{4}{5}$?

In what quadrants must an angle be taken whose cosecant $= -2$?

3. In what quadrants must A, A_1, &c., be taken in each of the following cases?

$\sin A = +0.809 \qquad \cos A_1 = -0.599$

$\tan A_2 = -\sqrt{\tfrac{1}{3}} \qquad \text{ctn}\, A_3 = +0.466$

$\sec A_4 = -\sqrt{2} \qquad \csc A_5 = +3.86$

4. Which of the trigonometric functions of an angle are never numerically less than unity? which never numerically greater than unity? which sometimes greater and sometimes less than unity?

5. If $\csc A = \sec B$, what is the relation between A and B? Give all the possible answers.

6. Given $\sin A = \tfrac{1}{2}\sqrt{3}$, and $\cos A = \tfrac{1}{2}$; find all the other functions of A.

7. Given $\cos A = -\sqrt{\tfrac{1}{2}}$; find all the other functions of A when A is an angle of the 1st quadrant; find also all the functions of $90° - A$.

8. Given $\cos A = \dfrac{1}{2\sqrt{2}}$; find all the other functions of A when A is an angle of the 4th quadrant.

9. Given $\tan A = \sqrt{3}$; find all the other functions of A when A is an angle of the 3d quadrant.

10. Given $\text{ctn}\, A = \tfrac{3}{4}$; find all the possible values of other functions of A.

11. Prove $\sec^2 A \,.\, \csc^2 A = \sec^2 A + \csc^2 A$.

12. Draw lines in and about a circle whose radius is unity, which represent the functions of an angle A of the 2d quadrant when $\cos A = -\tfrac{1}{2}$; also when A is an angle of the 4th quadrant, and $\cos A = +\tfrac{1}{2}$.

13. Suppose A to be an angle of the third quadrant; draw lines as above when $\tan A = +1$.

14. Find by geometry all the functions of 45°. Given chord $36° = 0.618$, find sin 18°.

15. Given $\sin A = \frac{1}{2}$, and $A < 90°$, what are the functions of the angles $-A$, $90° \pm A$, $180° \pm A$, $270° \pm A$, $360° \pm A$?

16. Given $\sin 30 = \frac{1}{2}$; what are the functions of the angles 3270°, 2580°, 750°, −150°, −60°, −120°?

17. Show that formulas 8–15 hold when φ is an angle of the 2d quadrant.

18. The same as the above when φ is an angle of the 3d quadrant.

19. Find all the functions of the following angles:

210°, 225°, 240°, 300°, 315°, −45°.

20. What values of A, A_1, A_2, and A_3, between 0° and 720°, will satisfy the following equations: —

$$\sin A = +\tfrac{1}{2} \qquad \cos A_1 = -\sqrt{\tfrac{1}{2}}$$

$$\tan A_2 = +\sqrt{3} \qquad \tan^2 A_3 + 4 \sin^2 A_3 = 0$$

21. What angles less than 360° have a cosine $= -\sqrt{\frac{1}{2}}$? what a secant $= -2$? what a tangent $= +\sqrt{3}$?

22. Which of the following angles have a cosine $= -\frac{1}{2}$? which a tangent $= -1$? which a secant $= +\sqrt{2}$? 45°, 120°, 225°, 240°, 315°, −240°, −315°, 600°.

23. What values of A between 0° and 720° will satisfy the equation $\sin^2 A + 5 \cos^2 A = 3$?

24. Given $\sin^2 A = m \cos A - n$; find cos A.

25. Given $\sin A = m \sin B$, and $\tan A = n \tan B$; find sin A and cos B.

26. Show what change is taking place in each of the functions of an angle when the angle is increasing from 0° to 90°, from 90° to 180°, from 180° to 270°, and from 270° to 360°.

27. Given $\sin \theta + \cos \theta = 1$; find θ.

28. Given $\sin^2\theta = \frac{3}{2\sec\theta}$; find θ.

29. Given $\sin^2\theta - 2\cos\theta + \frac{1}{4} = 0$; find θ.

30. Prove:—

$\sin^2\theta \,.\, \tan\theta + \cos^2\theta \,.\, \text{ctn}\,\theta + 2\sin\theta\cos\theta = \tan\theta + \text{ctn}\,\theta$.

31. Given $\cos 288° = 0.309$; find $\text{ctn}\,160°$. (Carry the result to three places of decimals.)

32. Given $\tan 241° = 1.8$; find $\sin 299°$.

33. Given $\cos x = \frac{\cos A}{\sin C}$ and $\cos\,(90 - x) = \frac{\cos B}{\cos C}$;

Prove $\cos^2 A + \cos^2 B + \cos^2 C = 1$.

34. Given $A = \sin^{-1}\frac{3}{5}$, $B = \sin^{-1}\frac{4}{5}$;

Prove $A + B = 90°$.

35. Prove $\sin^{-1}x + \cos^{-1}x = 90°$.

36. Find the value of $\tan\,(\tan^{-1}x + \text{ctn}^{-1}x)$.

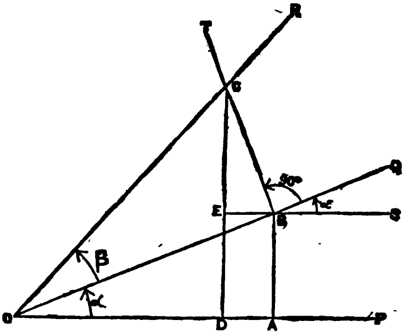
CHAPTER III.

FUNCTIONS OF THE SUM AND DIFFERENCE OF TWO ANGLES, AND FORMULAS DEDUCED FROM THEM.

§ **46.** *Given the sine and cosine of any two angles, to find the sine and cosine of their sum and of their difference.*

Let α and β denote the two angles, and let OP, OA, and OR be so drawn in a plane that $\alpha =$ POQ, $\beta =$ QOR.

FIG. 34.



Then, by § 16, —

$$\alpha + \beta = \text{POQ} + \text{QOR} = \text{POR}.$$

The angles α, β, and $\alpha+\beta$, are all positive and acute in Fig. 34; but our solution is applicable to all possible cases (*v.* § 47).

From any point, C, of the line OR, drop perpendiculars on OP and OQ, meeting OP and OQ at D and B respectively. From B drop a perpendicular on OP, meeting OP at A. Then the triangles OAB, OBC, and ODC, are triangles of reference for α, β, and $\alpha+\beta$, respectively; and we have—

$$\sin\alpha=\frac{AB}{OB},\quad \sin\beta=\frac{BC}{OC},\quad \sin(\alpha+\beta)=\frac{DC}{OC},$$

$$\cos\alpha=\frac{OA}{OB},\quad \cos\beta=\frac{OB}{OC},\quad \cos(\alpha+\beta)=\frac{OD}{OC}.$$

Through B draw BS parallel to OP, and meeting DC at E. Let S and Q be so taken that BS and BQ are positive; and let T be so taken on BC that QBT is a positive right angle. Then, by § 21 *h* and § 16,—

$$SBT=SBQ+QBT=90^\circ+\alpha;$$

and BEC is a triangle of reference for this angle. Hence, by [13],—

$$\sin(90^\circ+\alpha)=\cos\alpha=\frac{EC}{BC},$$

$$\cos(90^\circ+\alpha)=-\sin\alpha=\frac{BE}{BC}.$$

Again, by § 2 and § 3, *d*,—

$$DC=DE+EC=AB+EC,\quad OD=OA+AD=OA+BE.$$

Then we have—

$$\sin(\alpha+\beta)=\frac{DC}{OC}=\frac{AB}{OC}+\frac{EC}{OC},$$

$$\cos(\alpha+\beta)=\frac{\text{OD}}{\text{OC}}=\frac{\text{OA}}{\text{OC}}+\frac{\text{BE}}{\text{OC}}.$$

The common denominator of the fractions which compose the last members of these equations is the hypothenuse of the triangle OBC, while the numerators are legs of the triangles OAB and BEC. In order to express each of these fractions in terms of the sine and cosine of α and β, we will multiply its numerator and denominator by a side which is common to the two triangles to which the numerator and denominator belong. Thus we have —

$$\frac{\text{AB}}{\text{OC}}+\frac{\text{EC}}{\text{OC}}=\frac{\text{AB}}{\text{OB}}\cdot\frac{\text{OB}}{\text{OC}}+\frac{\text{EC}}{\text{BC}}\cdot\frac{\text{BC}}{\text{OC}};$$

$$\frac{\text{OA}}{\text{OC}}+\frac{\text{AE}}{\text{OC}}=\frac{\text{OA}}{\text{OB}}\cdot\frac{\text{OB}}{\text{OC}}+\frac{\text{BE}}{\text{BC}}\cdot\frac{\text{BC}}{\text{OC}};$$

or, substituting the values given on the last page,

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta, \quad [18, a]$$

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta. \quad [19, a]$$

Since the above solution is applicable to all values of α and β (*v.* § 47), we may replace β by $-\beta$.

But, by [11], —

$$\sin(-\beta)=-\sin\beta, \qquad \cos(-\beta)=\cos\beta.$$

Hence we have —

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta, \quad [18, b]$$

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta. \quad [19, b]$$

The four formulas proved above can be embodied in two formulas, as follows: —

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta, \qquad [18]$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta. \qquad [19]$$

where we are to use either the upper signs throughout or the lower signs throughout.

These two formulas, which are constantly used in mathematics, should be committed to memory.

§ **47.** After the student has thoroughly mastered the above solution as applied to any one case of the problem, such as that of Fig. 34, he should carefully re-examine it, with the view of satisfying himself that it is equally valid for any possible case. To assist him in making this examination, we append Figs. 35–38; and the student is advised to draw figures which shall represent still other cases. But it is important to observe that the generality of the solution does not depend on a complete analysis of cases, but on its being stated in a form which can be seen at every point to admit of universal application.

FIG. 35.

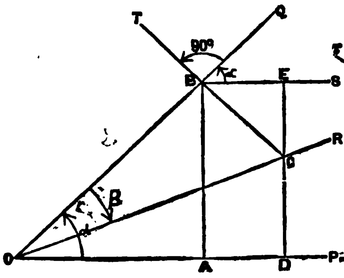


FIG. 36.

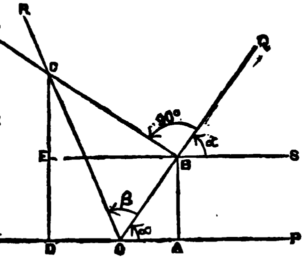


FIG. 37.

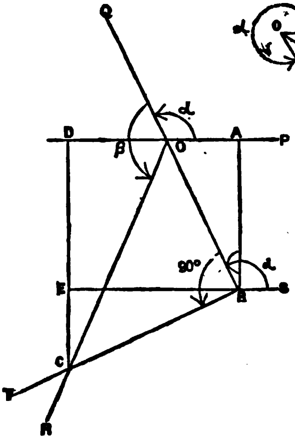
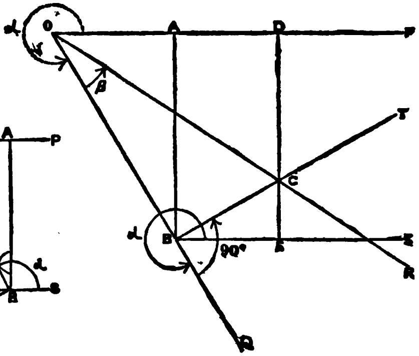


FIG. 38.



§ **48.** *Given the sine and cosine of the sum or difference of any two angles, to find the tangent and cotangent.*

By [2], [18], and [19], we have, —

$$\tan(\alpha \pm \beta) = \frac{\sin(\alpha \pm \beta)}{\cos(\alpha \pm \beta)} = \frac{\sin\alpha\cos\beta \pm \cos\alpha\sin\beta}{\cos\alpha\cos\beta \mp \sin\alpha\sin\beta}$$

Divide both numerator and denominator by $\cos\alpha\cos\beta$, and we get, —

$$\tan(\alpha \pm \beta) = \frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} \pm \dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta} \mp \dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta} \qquad [20]$$

By [1] and [20] we have, —

$$\frac{1}{\tan(\alpha \pm \beta)} = \operatorname{ctn}(\alpha \pm \beta) = \frac{1 \mp \tan\alpha\tan\beta}{\tan\alpha \pm \tan\beta}$$

Divide both numerator and denominator by $\tan\alpha\tan\beta$, and we get, —

$$\operatorname{ctn}(\alpha \pm \beta) = \frac{\operatorname{ctn}\alpha\operatorname{ctn}\beta \mp 1}{\operatorname{ctn}\beta \pm \operatorname{ctn}\alpha} \qquad [21]$$

§ 49. The following formulas are sometimes found useful. Let the student prove them by the aid of formulas [2, 18 *a*, 18 *b*, 19 *a*, 19 *b*, and 4.]

$$\frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}=\frac{\tan\alpha+\tan\beta}{\tan\alpha-\tan\beta}=\frac{\operatorname{ctn}\beta+\operatorname{ctn}\alpha}{\operatorname{ctn}\beta-\operatorname{ctn}\alpha}\qquad[22]$$

$$\frac{\cos(\alpha+\beta)}{\cos(\alpha-\beta)}=\frac{1-\tan\alpha\tan\beta}{1+\tan\alpha\tan\beta}=\frac{\operatorname{ctn}\beta-\tan\alpha}{\operatorname{ctn}\beta+\tan\alpha}\qquad[23]$$

$$\left.\begin{aligned}\sin(\alpha+\beta)\sin(\alpha-\beta)&=\sin^2\alpha-\sin^2\beta\\&=\cos^2\beta-\cos^2\alpha\end{aligned}\right\}\qquad[24]$$

$$\left.\begin{aligned}\cos(\alpha+\beta)\cos(\alpha-\beta)&=\cos^2\alpha-\sin^2\beta\\&=\cos^2\beta-\sin^2\alpha\end{aligned}\right\}\qquad[25]$$

§ 50. *Given the functions of any angle, to find the functions of twice that angle.*

Substituting in [18 *a*] and [19 *a*] α for β, we have,—

$$\left.\begin{aligned}\sin(\alpha+\alpha)=\sin 2\alpha&=\sin\alpha\cos\alpha+\cos\alpha\sin\alpha\\&=2\sin\alpha\cos\alpha\end{aligned}\right\}\qquad[26]$$

$$\left.\begin{aligned}\cos(\alpha+\alpha)=\cos 2\alpha&=\cos\alpha\cos\alpha-\sin\alpha\sin\alpha\\&=\cos^2\alpha-\sin^2\alpha\qquad\text{(I.)}\\&=2\cos^2\alpha-1\qquad\text{(II.)}\\&=1-2\sin^2\alpha\qquad\text{(III.)}\end{aligned}\right\}\qquad[27]$$

Substituting in [20] and [21] α for $\pm\beta$, we have,—

$$\tan(\alpha+\alpha)=\tan 2\alpha=\frac{\tan\alpha+\tan\alpha}{1-\tan\alpha\tan\alpha}=\frac{2\tan\alpha}{1-\tan^2\alpha}\qquad[28]$$

$$\operatorname{ctn}(\alpha+\alpha)=\operatorname{ctn}2\alpha=\frac{\operatorname{ctn}\alpha\operatorname{ctn}\alpha-1}{\operatorname{ctn}\alpha+\operatorname{ctn}\alpha}=\frac{\operatorname{ctn}^2\alpha-1}{2\operatorname{ctn}\alpha}\qquad[29]$$

§ **51.** *Given the functions of any angle to find the functions of half that angle.*

All the preceding formulas are perfectly general, and will apply as well to one angle as to another. We can therefore substitute $\frac{1}{2}\alpha$ throughout in place of α. Making this substitution, —

in [27 III.] we have, —

$$\cos\alpha = 1 - 2\sin^2\frac{\alpha}{2} \quad \therefore \sin\frac{\alpha}{2} = \sqrt{\tfrac{1}{2}(1-\cos\alpha)} \qquad [30]$$

in [27 II.] we have, —

$$\cos\alpha = 2\cos^2\frac{\alpha}{2} - 1 \quad \therefore \cos\frac{\alpha}{2} = \sqrt{\tfrac{1}{2}(1+\cos\alpha)} \qquad [31]$$

By [2], [30], and [31], —

$$\tan\frac{\alpha}{2} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = \sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} \qquad [32]$$

and by [3], —

$$\operatorname{ctn}\frac{\alpha}{2} = \frac{1}{\tan\frac{\alpha}{2}} = \sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}} \qquad [33]$$

§ **52.** By [29], —

$$\operatorname{ctn} 2\alpha = \frac{\operatorname{ctn}^2\alpha - 1}{2\operatorname{ctn}\alpha} = \frac{1}{2}\,\frac{\operatorname{ctn}^2\alpha - 1}{\operatorname{ctn}\alpha} = \tfrac{1}{2}(\operatorname{ctn}\alpha - \tan\alpha) \; [34]$$

By [1] and [26], —

$$\csc 2\alpha = \frac{1}{\sin 2\alpha} = \frac{1}{2\sin\alpha\cos\alpha} = \tfrac{1}{2}\sec\alpha\csc\alpha$$

$$= \text{by } [4]\ \frac{\sin^2\alpha + \cos^2\alpha}{2\sin\alpha\cos\alpha} = \tfrac{1}{2}(\tan\alpha + \operatorname{ctn}\alpha) \; [35]$$

By subtracting [34] from [35] we get,—

$$\tan\alpha = \csc 2\,\alpha - \operatorname{ctn} 2\,\alpha.$$

Now, substituting $\frac{1}{2}\,\alpha$ for α, we get,—

$$\tan\tfrac{1}{2}\,\alpha = \csc\alpha - \operatorname{ctn}\alpha. \qquad [36]$$

By adding [34] to [35], and substituting as above, we get,—

$$\operatorname{ctn}\tfrac{1}{2}\,\alpha = \csc\alpha + \operatorname{ctn}\alpha = \frac{1}{\csc\alpha - \operatorname{ctn}\alpha}. \qquad [37]$$

Substituting $45° + \frac{1}{2}\,\alpha$, and $45° - \frac{1}{2}\,\alpha$, for α in [36], and we get,—

$$\begin{aligned}\tan(45° + \tfrac{1}{2}\,\alpha) &= \csc(90° + \alpha) + \operatorname{ctn}(90° + \alpha)\\ &= \text{by [13] } \sec\alpha + \tan\alpha. \qquad [38]\end{aligned}$$

and

$$\tan(45° - \tfrac{1}{2}\,\alpha) = \csc(90° - \alpha) + \operatorname{ctn}(90° - \alpha)$$

$$= \operatorname{ctn}(45° + \tfrac{1}{2}\,\alpha) = \sec\alpha - \tan\alpha = \frac{1}{\sec\alpha + \tan\alpha} \qquad [39]$$

§ **53.** Some formulas will now be obtained, the usefulness of which will be seen hereafter.

By [18 *a*] and [18 *b*],—

$$\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin\alpha\cos\beta \qquad [40]$$

and $$\sin(\alpha + \beta) - \sin(\alpha - \beta) = 2\cos\alpha\sin\beta \qquad [41]$$

By [19 *a*] and [19 *b*],—

$$\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2\cos\alpha\cos\beta \qquad [42]$$

and $$\cos(\alpha - \beta) - \cos(\alpha + \beta) = 2\sin\alpha\sin\beta \qquad [43]$$

Put $\alpha+\beta=x \therefore 2\alpha=x+y \therefore \alpha=\frac{1}{2}(x+y)$

$\alpha-\beta=y \quad 2\beta=x-y \therefore \beta=\frac{1}{2}(x-y)$

Substituting these values of α and β in [42] and [43], we get, —

$$\sin x+\sin y=2\sin\tfrac{1}{2}(x+y)\cos\tfrac{1}{2}(x-y)$$

$$\sin x-\sin y=2\cos\tfrac{1}{2}(x+y)\sin\tfrac{1}{2}(x-y)$$

$$\cos x+\cos y=2\cos\tfrac{1}{2}(x+y)\cos\tfrac{1}{2}(x-y)$$

$$\cos y-\cos x=2\sin\tfrac{1}{2}(x+y)\sin\tfrac{1}{2}(x-y)$$

Since these last formulas are perfectly general, we can write any other letters, and therefore α and β, in place of x and y, as follows: —

$$\sin\alpha+\sin\beta=2\sin\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha-\beta) \qquad [44]$$

$$\sin\alpha-\sin\beta=2\cos\tfrac{1}{2}(\alpha+\beta)\sin\tfrac{1}{2}(\alpha-\beta) \qquad [45]$$

$$\cos\alpha+\cos\beta=2\cos\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha-\beta) \qquad [46]$$

$$\cos\beta-\cos\alpha=2\sin\tfrac{1}{2}(\alpha+\beta)\sin\tfrac{1}{2}(\alpha-\beta) \qquad [47]$$

By [44] and [45] we have, —

$$\left.\begin{aligned}\frac{\sin\alpha+\sin\beta}{\sin\alpha-\sin\beta}&=\frac{2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha+\beta)}{2\cos\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)}\\ &=\tan\tfrac{1}{2}(\alpha+\beta)\operatorname{ctn}\tfrac{1}{2}(\alpha-\beta)=\frac{\tan\frac{1}{2}(\alpha+\beta)}{\tan\frac{1}{2}(\alpha-\beta)}\end{aligned}\right\} \quad [48]$$

In like manner we get, —

$$\frac{\cos\alpha-\cos\beta}{\cos\alpha+\cos\beta}=-\tan\tfrac{1}{2}(\alpha+\beta)\tan\tfrac{1}{2}(\alpha-\beta) \qquad [49]$$

$$\frac{\sin\alpha\pm\sin\beta}{\cos\alpha+\cos\beta}=\tan\tfrac{1}{2}(\alpha\pm\beta) \qquad [50]$$

$$\frac{\sin\alpha\mp\sin\beta}{\cos\beta-\cos\alpha}=\operatorname{ctn}\tfrac{1}{2}(\alpha\pm\beta) \qquad [51]$$

EXAMPLES.

1. Find the sines and cosines of the following angles by the aid of formulas 18, 19, and 12: 0°, 90°, 180°, 270°, 360°, $(90^\circ \pm A)$ $(180^\circ \pm A)$ $(270^\circ \pm A)$, $(360^\circ \pm A)$, and $-A$. (To get sin 0°, put $\beta = \alpha$ in [18, *b*].)

2. From the functions of 30° find those of 15°.

3. Prove $\cos(60^\circ + A) + \cos(60^\circ - A) = \cos A$.

4. Prove $\sin(30^\circ + A) + \sin(30^\circ - A) = \cos A$.

5. Prove $\sin A = \dfrac{2 \tan \frac{1}{2} A}{1 + \tan^2 \frac{1}{2} A}$.

6. Find $\sin 3A$ in terms of $\sin A$.

7. Find $\cos 3A$ in terms of $\cos A$.

8. From the functions of 90° find those of 45°.

9. Given $\sin A = \frac{2}{5}$; find $\sin \frac{1}{2} A$; also $\cos \frac{1}{2} A$.

10. Given $2 \sin A = \sin 2A$; find A.

11. Given $\tan 2A = 3 \tan A$; find A.

12. Prove $\tan 50^\circ + \operatorname{ctn} 50^\circ = 2 \sec 10^\circ$.

13. Given $\sin 280^\circ = m$; find $\sin 160^\circ$.

14. Given $\sin \theta + \cos \theta = \dfrac{1}{\sqrt{2}}$; find θ.

15. Given $\tan(45^\circ - \theta) + \operatorname{ctn}(45^\circ - \theta) = 4$; find θ.

16. Prove $\dfrac{\cos A + \sin A}{\cos A - \sin A} = \tan 2A + \sec 2A$.

17. Prove

$$2 \sin^2 A \sin^2 B + 2 \cos^2 A \cos^2 B = 1 + \cos 2A \cos 2B.$$

18. Prove

$$\tan(45^\circ + A) - \tan(45^\circ - A) = 2 \tan 2A.$$

19. Prove $\dfrac{\sin B}{\sin A} = \dfrac{\sin(2A + B)}{\sin A} - 2\cos(A + B)$.

20. Prove $\dfrac{1 - \tan^2(45^\circ - A)}{1 + \tan^2(45^\circ - A)} = \sin 2A$.

21. Prove $\frac{4\tan A\,(1-\tan^2 A)}{(1+\tan^2 A)^2} = \sin 4A$.

22. Prove

$$\sin A\,(1+\tan A) + \cos A\,(1+\operatorname{ctn} A) = \csc A + \sec A.$$

23. Prove

$$\cos A + \cos\,(120^\circ - A) + \cos\,(120^\circ + A) = 0.$$

24. Prove

$$4 \sin A \sin\,(60^\circ - A)\,\sin\,(60^\circ + A) = \sin 3A.$$

25. Given $\sin 3\theta + \sin 2\theta + \sin\theta = 0$; find θ.

26. Given $\sin\,(x+\alpha) = \cos\,(x-\alpha)$; find x.

27. Prove $2\sec 2A = \sec\,(45^\circ + A)\,\sec\,(45^\circ - A)$.

28. Given $\tan\theta = \frac{b}{a}$; find the value of $a\cos 2\theta + b\sin 2\theta$.

In the next five formulas k denotes any integer. These formulas can be readily proved by making the proper substitutions in formulas 35, 36, 37, 38, and 21.

29. Prove

$$\sin k\alpha = 2\sin\,(k-1)\,\alpha\cos\alpha - \sin\,(k-2)\,\alpha.$$

30. Prove

$$\sin k\alpha = 2\cos\,(k-1)\,\alpha\sin\alpha + \sin\,(k-2)\,\alpha.$$

31. Prove

$$\cos k\alpha = 2\cos\,(k-1)\,\alpha\cos\alpha - \cos\,(k-2)\,\alpha.$$

32. Prove

$$\cos k\alpha = 2\sin\,(k-1)\,\alpha\sin\alpha + \cos\,(k-2)\,\alpha.$$

33. Prove $\tan k\alpha = \frac{\tan\,(k-1)\,\alpha + \tan\alpha}{1 - \tan\,(k-1)\,\alpha\tan\alpha}$.

34. Prove by the aid of the formula of Ex. 31 $\cos 3A = 4\cos^3 A - 3\cos A$.

35. Prove $\sqrt{1+\sin A} = 1 + 2\sin\frac{1}{4}A\sqrt{1-\sin\frac{1}{2}A}$, and show that A must be between 180° and 270°.

36. Prove $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}$.

37. Prove $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = 45°$.

38. Prove $\sin^{-1}\frac{1}{\sqrt{5}} + \cot^{-1}3 = 45°$.

39. Given $\theta = \cos^{-1}\sqrt{\frac{2}{3}}$, and $\varphi = \cos^{-1}\frac{\sqrt{3}+\sqrt{2}}{2\sqrt{2}}$; prove $\varphi + \theta = 60°$.

40. Given $\text{vers}^{-1}\frac{x}{a} - \text{vers}^{-1}\frac{bx}{a} = \text{vers}^{-1}(1-b)$; prove $\frac{x}{a} = \pm\sqrt{\frac{2b}{1+b}}$.

41. Obtain $\sin\frac{A}{2}$ in terms of sin A when $\frac{A}{2}$ lies between $-45°$ and $-135°$.

42. Find cos 105°.

43. Find tan 165°.

44. Determine the limits between which A must lie in order that

$$2\sin A = +\sqrt{1+\sin 2A} - \sqrt{1-\sin 2A}.$$

45. If $\tan\frac{\theta}{2} = \left(\frac{1+c}{1-c}\right)^{\frac{1}{2}}\tan\frac{\varphi}{2}$, show that $\cos\theta = \frac{\cos\varphi - c}{1 - c\cos\varphi}$.

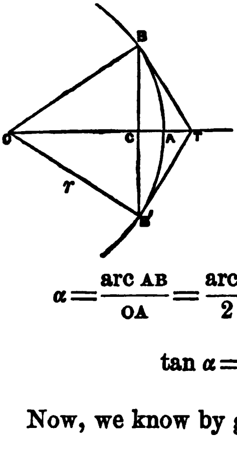
CHAPTER IV.

TRIGONOMETRIC TABLES.

§ 54. *Functions of small angles.*

Let AOB be any positive acute angle, and let α be its circular measure. Let AOB$'= -\alpha$, and therefore B$'$OB $= 2\alpha$; and let O be the centre of a circle of which the radius is $r =$ OA $=$ OB $=$ OB$'$. Let the chord BB$'$ cut OA at C; and let the tangents at B and B$'$ meet at T, which must lie in OA produced. Then, —

FIG. 39.



$$\alpha = \frac{\text{arc AB}}{\text{OA}} = \frac{\text{arc BB}'}{2r}, \qquad \sin\alpha = \frac{\text{CB}}{\text{OA}} = \frac{\text{chord BB}'}{2r},$$

$$\tan\alpha = \frac{\text{TB}}{\text{OB}} = \frac{\text{B}'\text{T}}{\text{OB}'} = \frac{\text{B}'\text{T} + \text{TB}}{2r}.$$

Now, we know by geometry, that

$$\text{chord BB}' < \text{arc BB}' < \text{B}'\text{T} + \text{TB}.$$

Hence, dividing through by $2r$, and substituting the above values, we have, —

$$\sin\alpha < \alpha < \tan\alpha; \qquad [52]$$

or, *the circular measure of an acute angle is greater than the sine of the angle, and less than the tangent of the angle.*

The last written inequality may assume the form, —

$$\frac{1}{\sin\alpha} > \frac{1}{\alpha} > \frac{1}{\tan\alpha}.$$

Now, multiplying through by sin α, we have —

$$1 > \frac{\sin\alpha}{\alpha} > \cos\alpha; \qquad [53]$$

and, multiplying through by tan α, —

$$\sec\alpha > \frac{\tan\alpha}{\alpha} > 1. \qquad [54]$$

But, if $\alpha \doteq 0$, $\cos\alpha = \sec\alpha = 1$; and the differences $1 - \cos\alpha$ (*v.* [53]) and $\sec\alpha - 1$ (*v.* [54]) can be made less than any assigned quantity by taking α sufficiently small. Hence, —

$$\frac{\sin\alpha}{\alpha} = 1 - \varepsilon, \quad \frac{\tan\alpha}{\alpha} = 1 + \varepsilon', \qquad [55]$$

where ε and ε' are positive quantities which decrease indefinitely when α decreases. We have then —

$$\sin\alpha = (1 - \varepsilon)\,\alpha, \quad \tan\alpha = (1 + \varepsilon')\,\alpha; \qquad [56]$$

or, if α is very small, —

$$\sin\alpha = \tan\alpha = \alpha, \text{ approximately.} \qquad [57]$$

For example, —

$$\sin 1' = \tan 1' = \frac{\pi}{10800} = 0.0002908882\ldots.$$

§ 55. We have, by [30], —

$$1 - \cos\alpha = 2\sin^2 \tfrac{1}{2}\alpha.$$

Now, by [55], —

$$\sin\tfrac{1}{2}\alpha = (1 - \varepsilon'') \times \tfrac{1}{2}\alpha = \tfrac{1}{2}(1 - \varepsilon'')\,\alpha;$$

$$\therefore\ 1 - \cos\alpha = \tfrac{1}{2}(1 - \varepsilon'')^2\alpha^2,$$

$$\text{or}\quad \frac{1 - \cos\alpha}{\alpha^2} = \tfrac{1}{2}(1 - \varepsilon'')^2 < \tfrac{1}{2}, \qquad [58]$$

$$\text{or}\quad \cos\alpha = 1 - \tfrac{1}{2}(1 - \varepsilon'')^2\alpha^2 > 1 - \tfrac{1}{2}\alpha^2. \qquad [59]$$

If α is very small, since ε'' decreases indefinitely with α, —

$$\cos\alpha = 1 - \tfrac{1}{2}\alpha^2, \text{ approximately.} \qquad [60]$$

For example, —

$$\cos 1' = 1 - \frac{0.0000000846\ldots}{2} = 0.999999958\ldots$$

§ 56. The formulas proved in the last two sections enable us to compute approximate values of the sine, cosine, and tangent of a very small angle; and the number of decimal places to which these values are correct depends on the smallness of the angle. Let us next seek the limits of error of these formulas, so that we may know in what cases we have a right to use them. Such limits may be found as follows: —

Since $\sin\alpha = 2\sin\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\alpha$, by [26], and since $\sin\tfrac{1}{2}\alpha > \tfrac{1}{2}\alpha\cos\tfrac{1}{2}\alpha$, by [53], —

$$\therefore\ \sin\alpha > \alpha\cos^2\tfrac{1}{2}\alpha.$$

But since, by [52], $\sin\tfrac{1}{2}\alpha < \tfrac{1}{2}\alpha$,

$$\therefore\ \cos^2\tfrac{1}{2}\alpha = 1 - \sin^2\tfrac{1}{2}\alpha > 1 - \tfrac{1}{4}\alpha^2;$$

$$\therefore\ \sin\alpha > \alpha(1 - \tfrac{1}{4}\alpha^2) = \alpha - \tfrac{1}{4}\alpha^3. \qquad [61]$$

Again, by [30], $\cos\alpha = 1 - 2\sin^2\frac{1}{2}\alpha$; and, by the formula just proved, —

$$\sin\tfrac{1}{2}\alpha > \tfrac{1}{2}\alpha - \tfrac{1}{32}\alpha^3;$$

$$\therefore \cos\alpha < 1 - 2\left(\tfrac{1}{2}\alpha - \tfrac{1}{32}\alpha^3\right)^2 < 1 - \tfrac{1}{2}\alpha^2 + \tfrac{1}{16}\alpha^4. \qquad [62]$$

For example,

$$\text{if } \alpha = \text{circ. meas. } 1' = \frac{\pi}{10800} = 0.000290888\ldots.,$$

$$\tfrac{1}{4}\alpha^3 = 0.000000000006\ldots.,$$

$$\tfrac{1}{16}\alpha^4 = 0.0000000000000004\ldots.;$$

so that the above methods are capable of giving values of sin 1′ accurate to *eleven* places of decimals, and of cos 1′ accurate to *fifteen* places of decimals.

By methods of the higher mathematics, we can establish still smaller limits of error; proving that

$$\left.\begin{array}{l} \sin\alpha > \alpha - \frac{1}{6}\alpha^3, \\ \cos\alpha < 1 - \frac{1}{2}\alpha^2 + \frac{1}{24}\alpha^4. \end{array}\right\} \qquad [63]$$

Now we easily compute

$\frac{1}{6}\alpha^3 < 0.00001$, if $\alpha < 0.03915 =$ circ. meas. $2^\circ\, 14'.6$,
$\frac{1}{24}\alpha^4 < 0.00001$, if $\alpha < 0.12447 =$ circ. meas. $7^\circ\, 7'.9$.

Hence, up to these limits respectively, we may use the formulas $\sin\alpha = \alpha$, $\cos\alpha = 1 - \frac{1}{2}\alpha^2$, with an error less than a unit in the fifth place of decimals.

§ 57. *To show how a table of trigonometric functions can be computed.*

It is necessary to find the functions of angles in the first quadrant only, as the functions of angles in the other quadrants are easily deduced from these. Since the sine of an angle is the cosine of its complement, a complete table of sines will also be a complete table of cosines; and from the sine and cosine of an angle the other functions are readily computed. We need only show, then, how to construct a complete table of sines for the first quadrant.

We may use the formula $\sin\alpha=\alpha$ for small values of α, up to a certain limit depending on the number of decimal places used in the table, and determined as in the last section. We may then proceed by the following method. By [40], —

$$\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta.$$

Let α be any angle, and let β be so small that we can use the formula [60], —

$$\cos\beta=1-\tfrac{1}{2}\beta^2$$

with an error far below the greatest error allowed in the table; for example, let $\beta=$ circ. meas. 1′. Then the above formula gives, —

$$\sin(\alpha+\beta)=2\sin\alpha-\sin(\alpha-\beta)-\beta^2\sin\alpha,$$

or

$$\sin(\alpha+\beta)-\sin\alpha=\sin\alpha-\sin(\alpha-\beta)-\beta^2\sin\alpha. \quad [64]$$

If, then, we know $\sin\alpha$ and $\sin(\alpha-\beta)$, we can compute $\sin(\alpha+\beta)$; then from $\sin(\alpha+\beta)$ and $\sin\alpha$, we can get $\sin(\alpha+2\beta)$; and so on, using the successive formulas, —

$$\sin(\alpha+\beta)-\sin\alpha=\sin\alpha-\sin(\alpha-\beta)-\beta^2\sin\alpha,$$

$$\sin(\alpha+2\beta)-\sin(\alpha+\beta)=\sin(\alpha+\beta)-\sin\alpha-\beta^2\sin(\alpha+\beta),$$

$$\sin(\alpha+3\beta)-\sin(\alpha+2\beta)=\sin(\alpha+2\beta)-\sin(\alpha+\beta)-\beta^2\sin(\alpha+2\beta), \text{ \&c.}$$

If $\beta =$ circ. meas. $1' = 0.00029\ldots$, $\beta^2 = 0.0000000846\ldots$, and, as this small number is a constant multiplier in the last term, computation by the above formulas is quite an easy process.

By the successive application of these formulas, we may find the sines of all angles of the quadrant at any small interval β; for example, 1′.

§ **58.** Given $\sin 20° = 0.34202$, $\sin 20°\,30' = 0.35021$; let the student compute the following values by the above formulas, assuming $\beta =$ circ. meas. 30′: —

$$\begin{aligned} \sin 21° &= 0.3584, \\ \sin 21°\,30' &= 0.3665, \\ \sin 22° &= 0.3746, \\ \sin 22°\,30' &= 0.3827, \\ \sin 23° &= 0.3907. \end{aligned}$$

§ **59.** When the values of the sine have been computed up to 60°, its values for the rest of the quadrant can be easily found by the formula, —

$$\sin(60° + \alpha) - \sin(60° - \alpha) = 2\cos 60° \sin\alpha = \sin\alpha,$$

or

$$\sin(60° + \alpha) = \sin(60° - \alpha) + \sin\alpha.$$

Again, we have found in §§ 38, 39, the values of the trigonometric functions for 30°, 45°, and 60°; and from these we can compute those of 22° 30′, 15°, 11° 15′, 7° 30′, 5° 37′.5, 3° 45′, &c. The results thus obtained may be used as tests of the computation made by the method of § 57.

§ **60.** The trigonometric functions are called *natural* trigonometric functions, to distinguish them from their logarithms, which are called *logarithmic* trigonometric functions.

Tables of the natural and logarithmic trigonometric functions have been constructed; and a thorough understanding of the use of these tables is necessary for performing trigonometric computations.

Since there is some variation in the arrangement of different tables, the student can better learn their use by referring to the explanations which accompany the tables, than by any explanations which could be given here.

Peirce's four-place tables will be used in the solution of problems in the following chapters.

§ **61.** *Interpolation.* A trigonometric table cannot contain the functions of *all* angles of the quadrant, but only of angles separated by a certain interval; for example, an interval of 1′ or of 10′. The functions of intermediate angles can be found by the same principle of *proportional parts* which is applied to the table of logarithms of numbers. We can prove the validity of this principle for the trigonometric functions as follows:—

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

Now, if β is very small, we can by [60] and [57], take $\cos\beta=1$, $\sin\beta=\beta$, and thus we have

$$\sin(\alpha+\beta)=\sin\alpha+\beta\cos\alpha, \text{ approximately,}$$

$$\cos(\alpha+\beta)=\cos\alpha-\beta\sin\alpha, \text{ approximately.}$$

Hence, if β and β' are two very small angles, —

$$\frac{\sin(\alpha+\beta)-\sin\alpha}{\sin(\alpha+\beta')-\sin\alpha}=\frac{\beta\cos\alpha}{\beta'\cos\alpha}=\frac{\beta}{\beta'}, \text{ approximately.}$$

$$\frac{\cos\alpha-\cos(\alpha+\beta)}{\cos\alpha-\cos(\alpha+\beta')}=\frac{\beta\sin\alpha}{\beta'\sin\alpha}=\frac{\beta}{\beta'}, \text{ approximately.}$$

So that, for very small variations of an angle, the variations of the sine and cosine are approximately proportional to the variations of the angle. Now, it is a known principle of logarithms, that, for very small variations of a number, the variations of the logarithm are approximately proportional to those of the number (*Peirce's Elements of Logarithms*, § 15). Hence, for very small variations of an angle, the variations of the log sin and log cos are approximately proportional to those of the angle.

The same principle can be extended, by similar proofs, to the other natural and logarithmic trigonometric functions. The *limits* within which it may be properly employed can also be investigated by means of the formulas of §§ 55 and 56.

§ **62.** It must be remembered that tables of logarithms and of trigonometric functions only give the values of the functions to a certain number of decimal places. For example: log sin 16° is given by our four-place tables as 9.4403; but this only informs us that the true value lies somewhere between 9.44025 and 9.44035. Thus every logarithm (and every natural function as well) taken from a four-place table is liable to an error not exceeding ±0.00005; and, if the function is found by interpolation, its total error may have any value short of 0.0001. If, then, a logarithm is obtained by adding or subtracting several logarithms, the error of the resulting logarithm *may* be several times 0.00005.

"When a change of 0.0001 in any logarithmic trigonometric function produces a change of more than 1′ in the corresponding angle, the *four-place logarithm does not determine the angle to minutes*, since the uncertainty of ± 0.00005 in the logarithm will leave an uncertainty of ± 0′.5 in the angle. This will never be the case with the log tan or log ctn.

"It is to be observed, that, with four-place logarithms, the angle can be determined to minutes from its *log tan* or *log ctn in all parts of the quadrant; from its log sin or log csc* in the *first half* of the quadrant; and from its *log cos* or *log sec* in the *last half.* This observation will often direct the computer in the choice of his method, where several lie open to him."—*Peirce's Elements of Logarithms.*

CHAPTER V.

SOLUTION OF RIGHT TRIANGLES.

§ **63.** To *solve* a triangle is to obtain from its given parts the values of its unknown parts. In every triangle there are six parts; namely, three angles and three sides.

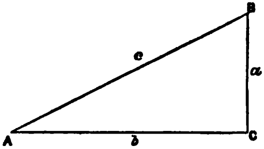
We have learned from geometry, that, in general, it is possible to *construct* a triangle when any three of its parts are given, provided one of them is a side. We shall now show, that, in general, it is also possible to solve a triangle when we have the above data.

In order to solve a triangle, we must obtain formulas by which each unknown part can be obtained from the known parts.

As we proceed, we shall find that there are often several formulas by which any particular part may be found. A formula should be selected which is adapted to *logarithmic computation*, and which will give accurate results; and it is also well, when possible, to use formulas by which each unknown part can be obtained directly from those given at the outset.

§ **64.** "After a table of logarithms has once been constructed, the labor of certain arithmetic operations can be materially diminished, while at the same time the chance of committing errors is lessened. The sole arithmetical operations which can be performed by logarithms are those of multiplication, division, involution, and evolution. Before, therefore, the value of an expression can be calculated by means of logarithms, the expression must be put into such a form that no other arithmetic operations than these have to be performed. Such an arrangement of an expression is called the *adaptation* of it to *logarithmic* computation." — *Snowball.*

FIG. 40.



Thus, when in a right triangle (Fig. 40) a and c are given, and b is required, logarithms cannot be *directly* applied to determine the value of b from the equation $b^2 = c^2 - a^2$ or $b = \sqrt{c^2 - a^2}$ (§ 21, e); but if the equation is put into the following form, $b = \sqrt{(c + a)(c - a)}$, logarithms can be directly applied to determine the value of b.

When a and b are given, and c is required, the formula $c^2 = a^2 + b^2$ not being adapted to logarithmic computation, will not, in general, be used to determine c, but the problem will be solved in a manner to be explained hereafter.

Since, as has been stated above, it is generally advantageous to make use of logarithms in problems involving arithmetic operations, care will be taken in this, and in the following chapter, to deduce and make use of formulas adapted to logarithmic computation.

§ **65.** When we know that one of the angles of a triangle is 90°, i.e., that the triangle is a right triangle, we can solve it by simpler methods than when all of the angles are oblique; we shall therefore consider right triangles separately, although they can be solved perfectly well by the general formulas which will be obtained hereafter.

When, in addition to the right angle, a side and one other part of a right triangle are given, it can be solved by formulas with which the student is already familiar, but which, for the sake of convenience, will be collected below.

$$A + B = 90° \quad \text{(I.)}$$

$$\left.\begin{array}{l}\sin A = \frac{a}{c}\\ \csc A = \frac{c}{a}\end{array}\right\}\text{(II.)} \qquad \left.\begin{array}{l}\cos A = \frac{b}{c}\\ \sec A = \frac{c}{b}\end{array}\right\}\text{(III.)} \qquad \left.\begin{array}{l}\tan A = \frac{a}{b}\\ \text{ctn}\, A = \frac{b}{a}\end{array}\right\}\text{(IV.)}$$

$$c^2 = a^2 + b^2 \quad \left\{\begin{array}{l}c^2 - a^2 = b^2 = (c+a)(c-a)\\ c^2 - b^2 = a^2 = (c+b)(c-b)\end{array}\right.$$

$$\therefore \begin{array}{l} b = \sqrt{(c+a)\ (c-a)}\\ a = \sqrt{(c+b)\ (c-b)}\end{array} \qquad [65]$$

§ 66. It will be worth while to begin by considering what cases can arise in the solution of right triangles. One side must be given, and may be either a leg or the hypothenuse. If a leg is given, there may be joined with it either the other leg, the hypothenuse, or an acute angle. If the hypothenuse is given, there may be joined with it either a leg (a case already considered) or an acute angle.

The following, then, are all the cases which can arise: —

CASE I. Given the two legs.
CASE II. Given a leg and the hypothenuse.
CASE III. Given a leg and an acute angle.
CASE IV. Given the hypothenuse and an acute angle.

In each of these cases, any one of the required parts may be obtained from the given parts by using that one of the formulas in § 65 *which involves the parts in question.*

The general rule given above should be rigorously followed, except in those cases where it will give formulas which are not adapted to logarithmic computation, or which will not give accurate results.

§ **67.** Case I. When we wish to find the hypothenuse from the two legs, the formula which we should choose in accordance with the rule, — i.e., $c^2 = a^2 + b^2$, — is not adapted to logarithmic computation. Here, then, we we shall have to deviate from the rule; and, in finding the hypothenuse, we will proceed as follows: we will first find the angles by the formula, $\tan \mathrm{A} = \operatorname{ctn} \mathrm{B} = \frac{a}{b}$; and having found the angles we will get c from the formula, $\csc \mathrm{A} = \frac{c}{a}$, or $\csc \mathrm{B} = \frac{c}{b}$.

If either angle is very small (say less than 10°), it is best to employ the csc of the *larger* angle; since a small angle must be given with great precision in order to determine accurately its sine and cosecant.

Example I. Given $a = 643.6$, $b = 59.87$.

$$\log \tan \mathrm{A} = \log a + \operatorname{colog} b = \log \operatorname{ctn} \mathrm{B},$$
$$\log c \quad = \log a + \log \csc \mathrm{A},$$

$$\begin{array}{lll}
\log \quad a = 2.8086 & & \log \quad a = 2.8086 \\
\operatorname{colog} \quad b = \underline{8.2228} & & \log \csc \mathrm{A} = \underline{0.0019} \\
\log \tan \mathrm{A} = 1.0314 = \log \operatorname{ctn} \mathrm{B}, & & \log \quad c = 2.8105, \\
\therefore \quad \mathrm{A} = 84^\circ\, 41', \ \mathrm{B} = 5^\circ\, 19' & & c = 646.4
\end{array}$$

Example II. Given $a = 0.0227$, $b = 0.0840$.

Answers. $\mathrm{A} = 15^\circ\, 7'$; $\mathrm{B} = 74^\circ\, 53'$; $c = 0.0870$.

§ **68.** Case II. Given a *leg and the hypothenuse* (c), we will denote the given leg by b.

Proceeding in accordance with the rule, we shall get the formulas

$$\sin \mathrm{B} = \cos \mathrm{A} = \frac{b}{c}, \text{ and } a = \sqrt{(c+b)(c-b)},$$

by which the following examples can be solved: —

Example I. Given $c = 7.458$, $b = 0.6473$.

Answers. $a = 7.430$, $\mathrm{A} = 85° \, 1'$, $\mathrm{B} = 4° \, 59'$.

Example II. Given $c = 672.3$, $b = 548.9$.

Answers. $a = 388.2$, $\mathrm{A} = 35° \, 17'$, $\mathrm{B} = 54° \, 43'$.

The above method is the best when b differs considerably from c; but, when b is nearly equal to c, sin B and cos A will be nearly equal to unity; B will therefore be nearly equal to 90°, and A will be nearly equal to 0°. When this is the case neither A nor B can be obtained with great accuracy from the formula $\sin \mathrm{B} = \cos \mathrm{A} = \frac{b}{c}$ (*v.* § 62 end).

We will now deduce a new formula by which, when b is nearly equal to c, we can find A (and therefore its complement B) more accurately than by the above method.

From [30], —

$$2 \sin^2 \tfrac{1}{2} \mathrm{A} = 1 - \cos \mathrm{A} = 1 - \frac{b}{c} = \frac{c-b}{c},$$

$$\therefore \sin \tfrac{1}{2} \mathrm{A} = \sqrt{\frac{c-b}{2c}} \qquad [66]$$

Example III. Given $b = 7.4169$, $c = 7.4451$.

From [66], —

$$\log \sin \tfrac{1}{2} \mathrm{A} = \tfrac{1}{2} [\log (c-b) + \operatorname{colog} 2c].$$

The formula for finding a is, in accordance with the rule, $a=\sqrt{(c+b)\,(c-b)}$.

$$\therefore\ \log a=\tfrac{1}{2}\left[\log(c+b)+\log(c-b)\right]$$

$$\begin{array}{rrr}
2c=14.8902 & c+b=14.8620 & \log(c+b)=1.1722 \\
\log 2c=1.1730 & c-b=\ \ 0.0282 & \log(c-b)=\bar{2}.4502 \\
\cline{1-1}\cline{3-3}
\log(c-b)=\bar{2}.4502 & & 2\,|\,\underline{\bar{1}.6224} \\
\text{colog } 2c=8.8270 & & \log a=\bar{1}.8112 \\
\cline{1-1}
2\,|\,\underline{17.2772} & & a=0.6474 \\
\log\sin\tfrac{1}{2}A=8.6386 & &
\end{array}$$

$\tfrac{1}{2}A=2°\ 29'$, $A=4°\ 59'$, $90°-A=B=85°\ 1'$.

Example IV. Given $c=84.325$, $b=84.282$.

Answers. $a=2.692$, $A=1°\ 50'$, $B=88°\ 10'$.

§ **69.** CASE III. *Given a leg and an acute angle.*

In this case and the following case, the unknown angle is the complement of the given angle; while the unknown sides may be found by employing those functions of the given angle of which the numerators are the unknown sides, and the denominator the given side.

Example I. Given $b=0.084$, $A=15°\ 9'$.

Answers. $c=0.0870$, $a=0.0227$, $B=74°\ 51'$.

Example II. Given $a=64.82$, $A=10°\ 3'$.

Answers. $c=371.4$, $b=365.8$, $B=79°\ 57'$.

§ **70.** CASE IV. *Given the hypothenuse and an acute angle.*

Example I. Given $c=426.7$, $A=34°\ 15'$.

Answers. $a=240.1$, $b=352.7$, $B=55°\ 45'$.

Example II. Given $c=371.4$, $A=10°\ 3'$.

Answers. $B=79°\ 57'$, $b=365.8$, $a=64.82$.

EXAMPLES.

1. Given $c = 0.8423$ $A = 42°.28'$ $C = 90°$ solve
2. “ $c = 0.4497$ $B = 51° 25'.1$ $C = 90°$ “
3. “ $a = 984.2$ $A = 22° 59'$ $C = 90°$ “
4. “ $b = 6.732$ $A = 79° 13'$ $C = 90°$ “
5. “ $a = 31.$ $B = 86° 18'$ $C = 90°$ “
6. “ $b = 984.2$ $B = 22° 59'$ $C = 90°$ “
7. “ $c = 73$ $b = 55$ $C = 90°$ “
8. “ $c = 9$ $b = 8$ $C = 90°$ “
9. “ $c = 1006.$ $a = 1000$ $C = 90°$ “
10. “ $c = 86$ $a = 17$ $C = 90°$ “
11. “ $a = 0.001$ $b = 1.069$ $C = 90°$ “
12. “ $a = 272$ $b = 224$ $C = 90°$ “

13. Given one leg equal to .7, and the angle opposite this leg equal to $\frac{2}{3}$ the angle opposite the other leg: solve.

14. If one leg of a right triangle is $\frac{7}{8}$ the hypothenuse, what are the angles of the triangle? and what is the ratio of the other leg to the hypothenuse?

15. Given the perimeter and the perpendicular from the right angle on to the hypothenuse: find the three sides.

16. Having measured a distance of 200 feet in a direct horizontal line from the bottom of a steeple, the angle of elevation of its top was found to be 46° 30′: find the height of the steeple.

17. A river whose breadth AC is 200 feet runs at the foot of a tower CB which subtends an angle BAC of 25° 10′ at the edge of the bank: find the height of the tower.

18. A person on the top of a tower 50 feet high observes the angle of depression of two objects on the horizontal plane, which are in the same straight line with the tower, to be 30° and 45°: find their distances from each other and from the observer.

19. A tower 150 feet high throws a shadow 75 feet long upon the horizontal plane on which it stands: find the sun's altitude.

20. A tower stands by a river. A person on the opposite bank finds its elevation to be 60°; he recedes 40 yards in a direct line from the tower, and then finds the elevation to be 50°: find the breadth of the river.

21. A rope is fastened to the top of a building 60 feet high. The length of the rope is 109 feet: find the angle at which it is inclined to the horizon.

22. Standing straight in front of a house, opposite one corner, I find that its length subtends an angle whose tangent is 2, while its height subtends an angle whose tangent is $\frac{3}{8}$; the height of the house is 45 feet: find its length.

23. Given the area of a right triangle $= m$, and the hypothenuse $= n$: find formulas for the two legs and the two angles. E. g., $m = 30$, $n = 13$.

In what case is this problem impossible?

24. Given an angle and the area: find formulas for the three sides. E. g. $A = 38° 27'$, area 158.4.

25. Prove that in a right triangle, area $= s\,(s - c)$, where C is the right angle, and $s = \frac{1}{2}\,(a + b + c)$.

26. Wishing to know the height of an inaccessible hill, I took the angle of elevation of its top to be 60°; I then measured 100 feet away from the hill, and found the angle of elevation to be 45°: what is the height of the hill?

27. Find the angle which a flagstaff 5 yards long, and standing on the top of a tower 200 yards high, subtends at a point in the horizontal plane 100 yards from the base of the tower.

28. A flagstaff 20 feet high stands on a wall 40 feet high; at a point E, on a level with the bottom of the wall, the flagstaff subtends an angle of 10°: find the distance of E from the wall.

29. On the bank of a river there is a column 200 feet high, supporting a statue 30 feet high; the statue to an observer on the opposite bank subtends the same angle as a man 6 feet high standing at the base of the column: find the breadth of the river.

30. The height of a house subtends a right angle at an opposite window, the top being 60° above a horizontal line: find the height, when the breadth of the street is 30 feet.

CHAPTER VI.

FORMULAS RELATING TO OBLIQUE TRIANGLES. — SOLUTION OF OBLIQUE TRIANGLES.

§ **71.** *The sides of any triangle are proportional to the sines of the opposite angles.*

FIG. 41.

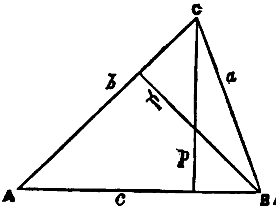
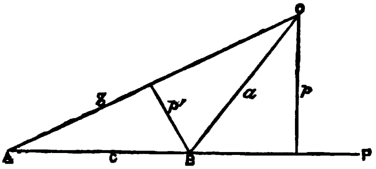


FIG. 42.



From B and C (Figs. 41 and 42) let fall the perpendiculars p' and p on b and c respectively. Then

$$\sin A = \frac{p}{b},\ \sin B = \frac{p}{a}\ \therefore\ \frac{\sin A}{\sin B} = \frac{p}{b} \times \frac{a}{p} = \frac{a}{b}, \qquad (\alpha)$$

and $$\sin A = \frac{p'}{c},\ \sin C = \frac{p'}{a}\ \therefore\ \frac{\sin A}{\sin C} = \frac{p'}{c} \times \frac{a}{p'} = \frac{a}{c}. \qquad (\beta)$$

From (α) we get, $\frac{a}{\sin A} = \frac{b}{\sin B}$, From ($\beta$) we get, $\dfrac{a}{\sin A}=\dfrac{c}{\sin C}$,

$$\therefore \frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}. \qquad [67]$$

In applying the above to Fig. 42, we must remember that sin B is equal to the sine of its supplement CBP.

§ **72.** *The sum of any two sides of a triangle is to their difference as the tangent of half the sum of the opposite angles is to the tangent of half their difference.*

From [67] we get, by the theory of proportions, —

$$\frac{\sin A+\sin B}{\sin A-\sin B}=\frac{a+b}{a-b}=\text{by } [48]\,\frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}, \quad [68]$$

In like manner

$$\frac{a+c}{a-c}=\frac{\tan\frac{1}{2}(A+C)}{\tan\frac{1}{2}(A-C)}, \text{ and } \frac{b+c}{b-c}=\frac{\tan\frac{1}{2}(B+C)}{\tan\frac{1}{2}(B-C)},$$

§ **73.** *The square of any side of a triangle is equal to the sum of the squares of the other two sides, minus twice the product of those sides into the cosine of their included angle.*

FIG. 43.

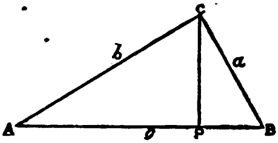
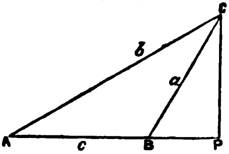


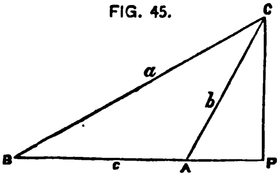
FIG 44.



Through C in the triangle ABC (Figs. 43, 44, and 45) draw PC perpendicular to c. In each figure, then,

$$a^2=PB^2+PC^2,$$

FIG. 45.



but $$\text{PB} = \text{PA} + c = -\text{AP} + c \quad (v.\ \S\ 3,\ d.)$$

$$\therefore \text{PB}^2 = \text{AP}^2 - 2\,\text{AP}\,c + c^2,$$

$$\therefore a^2 = \text{PC}^2 + \text{AP}^2 + c^2 - 2\,\text{AP}\,c,$$

but $\cos A = \frac{\text{AP}}{b}$,

$$\therefore \text{AP} = b \cos A, \text{ and } \text{PC}^2 + \text{AP}^2 = b^2.$$

$$\left.\begin{aligned} \therefore a^2 &= b^2 + c^2 - 2\,bc \cos A, \\ &\text{and in like manner} \\ b^2 &= c^2 + a^2 - 2\,c a \cos B, \\ c^2 &= a^2 + b^2 - 2\,a b \cos C. \end{aligned}\right\} \qquad [69]$$

Corollary. *Formula for the diagonal of a parallelogram.*

Let w denote an angle of a parallelogram, m and n its sides, d the diagonal which divides w, and w_1 the supplement of w.

By constructing a proper figure it can be readily seen that d, m, and n form a triangle in which the angle included by m and n is w_1.

$\therefore$ by [69], —

$$d^2 = m^2 + n^2 - 2\,m\,n \cos w_1\,;$$

but since w and w_1 are supplements, we have, —

$$\cos w_1 = -\cos w\,;$$

$$\therefore d^2 = m^2 + n^2 + 2\,m\,n \cos w. \qquad [70]$$

§ **74.** *Formula for the side of a triangle in terms of the cosines of the adjacent angles and the other two sides.*

From Figs. 43–45, it is evident that, —

$$\left.\begin{aligned} a &= b \cos C + c \cos B, \\ b &= c \cos A + a \cos C, \\ c &= a \cos B + b \cos A. \end{aligned}\right\} \qquad [71]$$

From [71] all the relations between the six parts of a triangle can be deduced by algebraic transformations; but the processes are somewhat longer than those given in the text.

§ 75. *Formulas for sine and cosine of* $\frac{1}{2}$A, $\frac{1}{2}$B, *and* $\frac{1}{2}$C. From [69] we find

$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \text{by [27 III.] } 1 - 2\sin^2\tfrac{1}{2}A,$$

$$\therefore 2\sin^2\tfrac{1}{2}A = 1 - \frac{b^2+c^2-a^2}{2bc} = \frac{a^2-b^2-c^2+2bc}{2bc},$$

$$= \frac{a^2-(b^2-2bc+c^2)}{2bc} = \frac{(a-b+c)(a+b-c)}{2bc},$$

put $2s = a+b+c$, $\therefore a-b+c = 2(s-b)$

and $a+b-c = 2(s-c)$

$$\therefore 2\sin^2\tfrac{1}{2}A = \frac{2(s-b)\,2(s-c)}{2bc}$$

$$\left.\begin{aligned} &\therefore \sin\tfrac{1}{2}A = +^{*}\sqrt{\frac{(s-b)(s-c)}{bc}},\\ &\text{and in like manner}\\ &\sin\tfrac{1}{2}B = +\sqrt{\frac{(s-c)(s-a)}{ca}},\\ &\sin\tfrac{1}{2}C = +\sqrt{\frac{(s-a)(s-b)}{ab}}, \end{aligned}\right\} \qquad [72]$$

Also, —

$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \text{by [27 II.] } 2\cos^2\tfrac{1}{2}A - 1,$$

---

[*] A, B, and C are less than 180°, and $\frac{1}{2}$ A, $\frac{1}{2}$ B, and $\frac{1}{2}$ C are less than 90°: we must then give the + sign to the radicals.

$$\therefore \cos^2\tfrac{1}{2}A = \frac{b^2+c^2-a^2+2bc}{4bc} = \frac{(b+c)^2-a^2}{4bc},$$

$$= \frac{(b+c+a)(b+c-a)}{4bc} = \frac{2s\,2(s-a)}{4bc} = \frac{s(s-a)}{bc}$$

$$\left.\begin{aligned} &\therefore \cos\tfrac{1}{2}A = +^{*}\sqrt{\frac{s(s-a)}{bc}}, \\ &\text{and in like manner} \\ &\cos\tfrac{1}{2}B = +\sqrt{\frac{s(s-b)}{ca}}, \\ &\cos\tfrac{1}{2}C = +\sqrt{\frac{s(s-c)}{ab}}, \end{aligned}\right\} \qquad [73]$$

Since the sine of an angle and the sine of its supplement are the same (*v.* [8]), whenever all that is given concerning an angle is the value of its sine, the angle may have either of two supplementary values. The ambiguity thus arising in the use of [72] is, however, removed by the consideration, that, since A, B, and C, being angles of a triangle, are each less than 180°, ½ A, ½ B, and ½ C are each less than 90°.

§ **76.** *Formulas for tan ½ A, tan ½ B, and tan ½ C, in terms of s, a, b, and c.*

By [2], [72], and [73] —

$$\tan\tfrac{1}{2}A = \frac{\sin\tfrac{1}{2}A}{\cos\tfrac{1}{2}A} = \frac{\sqrt{\dfrac{(s-b)(s-c)}{bc}}}{\sqrt{\dfrac{s(s-a)}{bc}}}$$

---

[*] *v.* note p. 75.

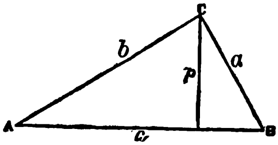
$$\therefore \tan \tfrac{1}{2} \mathrm{A} = +^{*}\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

and in like manner

$$\left.\begin{aligned} \tan \tfrac{1}{2} \mathrm{B} &= +\sqrt{\frac{(s-c)(s-a)}{s(s-b)}} \\ \tan \tfrac{1}{2} \mathrm{C} &= +\sqrt{\frac{(s-a)(s-b)}{s(s-c)}} \end{aligned}\right\} \qquad [74]$$

§ **77.** *Formulas for the area of a triangle, in terms of two sides and the included angle; in terms of one side and the adjacent angles; and in terms of s and the three sides.*

FIG. 46.



From geometry we know that the area of a triangle is half the product of its base by its altitude.

$$\therefore \text{ area of ABC (Fig. 46)} = \tfrac{1}{2} p\, c,$$

but $\sin \mathrm{A} = \frac{p}{b}$ $\therefore p = b \sin \mathrm{A}$.

∴ denoting the area by K we have, —

$$\mathrm{K} = \tfrac{1}{2} b\, c \sin \mathrm{A}. \qquad [75]$$

By [67], —

$$b = \frac{a \sin \mathrm{B}}{\sin \mathrm{A}}, \text{ and } c = \frac{a \sin \mathrm{C}}{\sin \mathrm{A}}.$$

Substituting these values in [75], and noting that, since $\mathrm{A} = 180^\circ - (\mathrm{B} + \mathrm{C})$, $\sin \mathrm{A} = \sin (\mathrm{B} + \mathrm{C})$, we get, —

$$\mathrm{K} = \tfrac{1}{2} \frac{a^2 \sin \mathrm{B} \sin \mathrm{C} \sin \mathrm{A}}{\sin^2 \mathrm{A}} = \frac{a^2 \sin \mathrm{B} \sin \mathrm{C}}{\sin (\mathrm{B} + \mathrm{C})}. \qquad [76]$$

---

* *v.* note p. 75.

By [26], —

$$\sin A = 2 \sin \tfrac{1}{2} A \cos \tfrac{1}{2} A.$$

∴ substituting in [75] we get, —

$$K = \tfrac{1}{2} b c \, 2 \sin \tfrac{1}{2} A \cos \tfrac{1}{2} A.$$

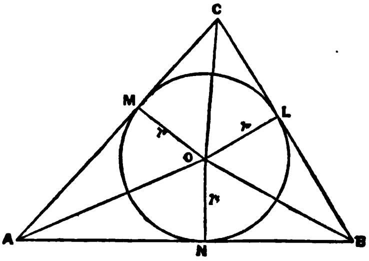
∴ by [72] and [73], —

$$K = b c \sqrt{\frac{(s-b)(s-c)(s-a)}{b c \cdot b c}}$$

$$= \sqrt{s(s-a)(s-b)(s-c)} \qquad [77]$$

§ **78.** *Formula for the area of a triangle, in terms of s and r (the radius of the inscribed circle).*

FIG. 47



Let O (Fig. 47) be the centre of the inscribed circle; then each of the lines OL, OM, and ON, perpendicular to a, b, and c respectively, will be equal to r.

Now, the triangle

$$ABC = BOC + COA + AOB = \tfrac{1}{2} a r + \tfrac{1}{2} b r + \tfrac{1}{2} c r.$$

$$\therefore\ K = \tfrac{1}{2}(a + b + c) r = s r \qquad [78]$$

§ **79.** *Formula for the radius of the inscribed circle, in terms of s, a, b, and c; and in terms of s,* A, B, *and* C.

By [78] and [77], —

$$r = \frac{K}{s} = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}. \qquad [79]$$

From [79] and [74] we get,—

$$r = s \tan \tfrac{1}{2} \mathrm{A} \tan \tfrac{1}{2} \mathrm{B} \tan \tfrac{1}{2} \mathrm{C}. \qquad [80]$$

§ **80.** *Formulas for* tan ½ A, tan ½ B, *and* tan ½ C, *in terms of* r, s, a, b, *and* c.

Dividing both sides of [79] by $s - a$, we get,—

$$\frac{r}{s-a} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

∴ by [74],—

$$\left.\begin{array}{l} \tan \tfrac{1}{2} \mathrm{A} = \dfrac{r}{s-a} \\ \text{and in like manner} \\ \tan \tfrac{1}{2} \mathrm{B} = \dfrac{r}{s-b} \\ \tan \tfrac{1}{2} \mathrm{C} = \dfrac{r}{s-c} \end{array}\right\} \qquad [81]$$

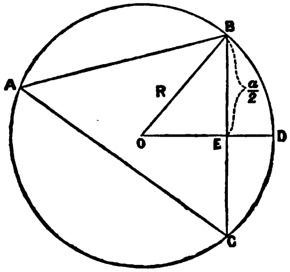
§ **81.** *Formulas for the perpendiculars* p_a, p_b, *and* p_c, *from the angles upon the sides* a, b, *and* c *respectively.*

$$\left.\begin{array}{l} p_a = b \sin \mathrm{C} = c \sin \mathrm{B} = \text{by [67] } \dfrac{bc}{a} \sin \mathrm{A} \\ \therefore \text{ by [75] } p_a = \dfrac{2\mathrm{K}}{a} \\ \text{and in like manner} \\ p_b = \dfrac{2\mathrm{K}}{b} \\ p_c = \dfrac{2\mathrm{K}}{c} \end{array}\right\} \qquad [82]$$

§ **82.** *Formulas for radius of circumscribed circle* (R), *in terms of a side and the opposite angle; and in terms of* K, *a*, *b*, *and c*.

In Fig. 48 draw the radius OD perpendicular to BC; therefore by geometry, —

FIG. 48.



$$\text{EB} = \tfrac{1}{2}\,\text{CB} = \tfrac{1}{2}\,a,$$

and
$$\text{arc DB} = \tfrac{1}{2}\,\text{arc CB};$$

$$\therefore \text{ the angle DOB} = \text{A}.$$

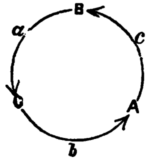
$$\therefore \csc \text{DOB} = \csc \text{A} = \frac{\text{R}}{\tfrac{1}{2}\,a}$$

$$\therefore \text{R} = \tfrac{1}{2}\,a \csc \text{A}. \qquad [83]$$

By [75], —

$$\sin \text{A} = \frac{2\,\text{K}}{b\,c} = \frac{1}{\csc \text{A}} \quad \therefore \csc \text{A} = \frac{b\,c}{2\,\text{K}}$$

$$\therefore \text{ by [83] } \text{R} = \frac{a\,b\,c}{4\,\text{K}} \qquad [84]$$



NOTE. — In each set of formulas numbered 67, 69, 71, 72, 73, 74, and 82, the first is perfectly general: from it, therefore, the other two can be obtained by direct substitution. The work of substitution is made easier by the following rule: Given the first in each set, the others are obtained by advancing the letters; that is, by substituting for each letter of the equation the next letter in the fixed order indicated by the figure in the margin. In this manner, the second of [74], for example, can be obtained from the first, and then

§ **83.** *Solution of oblique triangles.*

"We may remark here, that when an angle of a triangle is determined from its cosine, versed sine, tangent, cotangent, or secant, no uncertainty can exist about the angle, because only one angle exists less than 180° for which any of these functions has an assigned value. But when an angle of a triangle is determined from its sine or cosecant, uncertainty may exist, since there are two angles less than 180° which have a given sine or a given cosecant."—*Todhunter.*

There is, however, no uncertainty in the case of right triangles, because each angle, except the right angle, is acute

According to the methods of solution which we shall adopt, there is only one case of oblique triangles where there can be any uncertainty in the angle: this case will be fully discussed hereafter (*v.* Case II. § 85).

§ **84.** CASE I. *Given two angles and a side,—A, B, and a. find C, b, and c.*

$$A+B+C=180°, \therefore C=180°-(A+B).$$

the third from the second, and the first again from the third. In like manner, the second may be obtained from the third, the first from the second, and the third from the first, by moving the letters back.

By the above rule, we can get from [75], —

$$K=\tfrac{1}{2}ca \sin B,$$

and $$K=\tfrac{1}{2}ab \sin C.$$

Also from [83] we can get, —

$$R=\tfrac{1}{2}b \csc B,$$

and $$R=\tfrac{1}{2}c \csc C.$$

By [67],

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}, \therefore b=\frac{a\sin B}{\sin A}=a\sin B\csc A,$$

and $$c=\frac{a\sin C}{\sin A}=a\sin C\csc A;$$

$$\therefore \log b=\log a+\log\sin B+\log\csc A,$$

and $$\log c=\log a+\log\sin C+\log\csc A.$$

Example I.

Given $A=36°\ 56'$, $B=72°\ 6'$, $a=36.74$

$$\begin{array}{ll|ll} \log\csc A & =0.2212 & \log\csc A & =0.2212 \\ \log\sin B & =9.9784 & \log\sin C & =9.9759 \\ \log a & =1.5652 & \log a & =1.5652 \\ \hline \log b & =1.7648 & \log c & =1.7623 \\ b & =58.19 & c & =57.85 \end{array}$$

$A+B=109°\ 2'$ $\qquad$ $180°-(A+B)=C=70°\ 58'$

Example II.

Given $a=0.3578$, $B=32°\ 41'$, $C=47°\ 54'$.

Answers.

$C=47°\ 54'$, $b=0.1959$, $c=0.2691$.

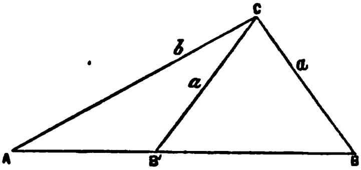
§ **85.** CASE II. *Given two sides and an angle opposite one of them,—a, b, and A: find c, B, and C.*

It has already been stated (§ 63), that, *in general*, a triangle can be solved when any three of its parts are given, provided one of the given parts is a side. It will now be shown, that, in the present case, the given parts can be so related to one another, that two different triangles can be obtained from them, that a single triangle can be obtained from them, and that no triangle can be obtained from them.

Suppose A to be less than 90°.

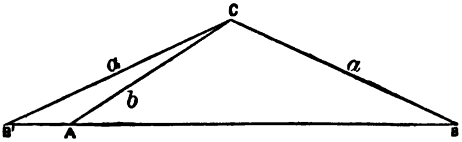
(1.) Make A (Fig. 49) equal to the given angle, and b equal to one of the given sides. From C as a centre, with a (the other given side) as a radius, describe an arc. If a is sufficiently large, and at the same time less than b, our arc will cut AB in two points, B and B′, both on the same side of A, and we shall have two triangles, ABC, AB′C, each of which will satisfy the conditions.

FIG. 49.



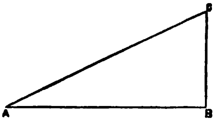
(2.) If a is equal to or greater than b, there will be only one triangle which will satisfy the conditions (*v.* Fig. 50).

FIG. 50.



(3.) If a is just equal to the perpendicular let fall from C to AB (Fig. 51), then the arc will cut the line AB in only one point, B, and we shall have only one triangle, namely, the right triangle ABC.

FIG. 51.



(4.) If a is less than BC (Fig. 50), then the arc does not cut the line AB, and there is therefore no triangle which will satisfy the conditions.

The length of the perpendicular BC (*v.* Fig. 51) is evidently b sin A.

The student will note that when there are two solutions (*v*. Fig. 49), we have, —

$$\text{CB}'\text{A} = 180° - \text{BB}'\text{C} = 180° - \text{CBA};$$

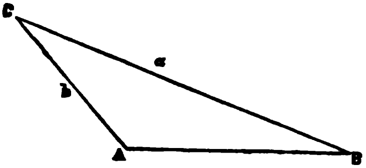
i.e., the two values of B are supplements.

When there is only one solution (Figs. 50 and 51), B cannot be greater than 90°.

Collecting the results of our discussion, we have, when A is acute, —

$$\left.\begin{array}{l}\text{Two solutions, when } a < b, \text{ and } a > b \sin A,\\ \text{One solution, when } a = b, \text{ or } a > b, \text{ or } a = b \sin A,\\ \text{No solution, when } a < b \sin A.\end{array}\right\} \quad [85]$$

FIG. 52.



When A = 90°, or A > 90°, we can never have two solutions, as can readily be seen from Fig. 52; and there will be no solution when

$$a < b, \text{ or } a = b.$$

In the present case, we have given, as has already been stated, a, b, and A.

By [67], —

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C};$$

$$\therefore \sin B = \frac{b \sin A}{a}$$

$$\therefore \log \sin B = \log b + \log \sin A + \text{colog } a,$$

$$C = 180° - (A + B).$$

$$c = \frac{a \sin C}{\sin A} \quad \therefore \log c = \log a + \log \sin C + \log \csc A.$$

Example I. Given $A = 32°$, $a = 988.1$, $b = 672.3$. A is acute, and $a > b$; therefore, by [85], we shall have only one solution, and B must be acute.

$$\begin{array}{rl@{\qquad\qquad}rl}
\log b &= 2.28276 & \log a &= 2.9948 \\
\log \sin A &= 9.7328 & \log \sin C &= 9.9096 \\
\text{colog } a &= \underline{7.0072} & \log \csc A &= \underline{0.2672} \\
\log \sin B &= 9.5656 & \log c &= 3.1716 \\
 & & c &= 1484
\end{array}$$

$$B = 21° \, 35', \; 180° - (A + B) = C = 125° \, 42'$$

Example II. Given $A = 48° \, 34$, $a = 46.24$, $b = 60.02$. A is acute, $a < b$, and $a > b \sin A$; therefore, by [85], we shall have two solutions; and the two values of B (B and B′) will be supplements of each other.

$$\begin{array}{rl@{\qquad\qquad}rl}
\text{colog } a &= 8.3350 & B &= 76° \, 43' \\
\log b &= 1.7783 & B' &= 103° \, 17 \\
\log \sin A &= \underline{9.8749} & C &= 54° \, 43' \\
\log \sin B &= 9.9882 & C' &= 28° \, 9'
\end{array}$$

$$\begin{array}{rl@{\qquad\qquad}rl}
\log a &= 1.6650 & \log a &= 1.6650 \\
\log \sin C &= 9.9119 & \log \sin C' &= 9.6737 \\
\log \csc A &= \underline{0.1251} & \log \csc A &= \underline{0.1251} \\
\log c &= 1.7020 & \log c' &= 1.4638 \\
c &= 50.35 & c' &= 29.10
\end{array}$$

Example III.

Given $a = 49.00$, $c = 31.24$, $C = 32° \, 18'$.

Answers.

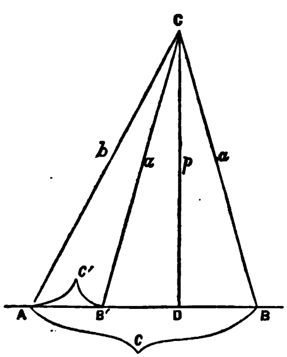
$A = 56° \, 57'$, $B = 90° \, 45'$, $b = 58 \, 46$.

$A' = 123° \, 03'$, $B' = 24° \, 39'$, $b' = 24.38$.

§ **86.** CASE II. *Second method.*

When the given angle, A, differs little from 90°, and when, at the same time, a is nearly equal to b, the angles B and B′ will differ little from 90°, and cannot, therefore, be determined with accuracy from their sines (*v.* § 62). In this case, the following method will give more accurate results: b and A being given, we can, by § 69, solve the right triangle ADC (Fig. 53), and find AD and p. Now, a and p being known, we can, by § 68, solve the triangle BDC, or its equal, B′DC, and find the angles B and BCD [$=$ B′CD], and the side DB.

FIG. 53.



From the above, we can find the required parts of the triangles ABC and AB′C, for

$$c = \mathrm{AD} + \mathrm{DB}, \quad c' = \mathrm{AD} - \mathrm{B'D},$$
$$C = \mathrm{ACD} + \mathrm{DCB} = 90^\circ - A + \mathrm{DCB},$$
$$C' = \mathrm{ACD} - \mathrm{DCB}\ [= \mathrm{B'CD}]$$

As has been stated before, when B and B′ fall on opposite sides of the point A, there will be only one solution; i.e., when B′D > AD, or when c' is negative.

Example. Given $A = 85^\circ\ 2'$, $a = 1076$, $b = 1078$. A is acute, $a < b$, and $a > b \sin A$: therefore, by [85], there are two solutions: —

1st, Solve, by § 69, the right triangle ADC (Fig. 53).

$$\begin{array}{rlrl} \log \sin A = & 9.9984 & \log \cos A = & 8.9374 \\ \log b = & \underline{3.0327} & \log b = & \underline{3.0327} \\ \log p = & 3.0311 & \log \mathrm{AD} = & 1.9701 \\ p = & 1074 & \mathrm{AD} = & 93.35 \end{array}$$

$$\mathrm{ACD} = 90 - A = 4^\circ\ 58'$$

2d, Solve, by § 68, the right triangle BDC.

Substituting, in [66], we get, —

$$\sin \tfrac{1}{2}\,\mathrm{DCB} = \sqrt{\frac{a-p}{2a}}$$

$$a - p = 2.$$
$$a + p = 2150.$$
$$2a = 2152.$$

$$\log (a-p) = 0.2010$$
$$\log (a+p) = 3.3324$$
$$2\,\underline{\overline{|\,3.6334}}$$
$$\log \mathrm{BD} = 1.8167$$
$$\mathrm{BD} = 65.57$$

$$\log (a-p) = 0.3010$$
$$\operatorname{colog} a = 6.9681$$
$$\operatorname{colog} 2 = 9.6990$$
$$2\,\underline{\overline{|\,16.9681}}$$
$$\log \sin \tfrac{1}{2}\,\mathrm{DCB} = 8.4840$$
$$\tfrac{1}{2}\,\mathrm{DCB} = 1^\circ\ 45'$$
$$\mathrm{DCB} = 3^\circ\ 30' = \mathrm{DCB}'$$

From the above we get —

$$\mathrm{ACB}' = 1^\circ\ 28' \qquad c' = 27.77$$
$$\mathrm{ACB} = 8^\circ\ 28' \qquad c = 158.9$$

§ 87. CASE III. *Given two sides and the included angle a, b, and* C.

$$\mathrm{A} + \mathrm{B} = 180^\circ - \mathrm{C}.$$

By [68] we have

$$\frac{a+b}{a-b} = \frac{\tan \frac{1}{2}(\mathrm{A}+\mathrm{B})}{\tan \frac{1}{2}(\mathrm{A}-\mathrm{B})},$$

$$\therefore\ (a+b) \tan \tfrac{1}{2}(\mathrm{A}-\mathrm{B}) = (a-b) \tan \tfrac{1}{2}(\mathrm{A}+\mathrm{B}),$$

$$\therefore\ \tan \tfrac{1}{2}(\mathrm{A}-\mathrm{B}) = \frac{a-b}{a+b} \tan \tfrac{1}{2}(\mathrm{A}+\mathrm{B}),$$

$$\therefore\ \log \tan \tfrac{1}{2}(\mathrm{A}-\mathrm{B}) = \log (a-b) + \log \tan \tfrac{1}{2}(\mathrm{A}+\mathrm{B}) + \operatorname{colog} (a+b).$$

We now have $\frac{1}{2}(A+B)$ and $\frac{1}{2}(A-B)$.

But $\frac{1}{2}(A-B)+\frac{1}{2}(A+B)=A$,

and $\frac{1}{2}(A+B)-\frac{1}{2}(A-B)=B$.

We now have to obtain c, which we do as in Cases I. and II.

By [67], —

$$\frac{a}{\sin A}=\frac{c}{\sin C}, \quad \therefore c=\frac{a \sin C}{\sin A},$$

$$\therefore \log c=\log a+\log \sin C+\log \csc A.$$

Example I.

Given $a=48.72$, $b=3984$, $C=68^\circ\ 34'$.

$$A+B=180-C=111^\circ\ 26',$$

$$\tfrac{1}{2}(A+B)=55^\circ\ 43.$$

$$\begin{array}{rlrr}
 & a+b=52.704 & \operatorname{colog}(a+b)= & 8.2782 \\
 & a-b=44.736 & \log(a-b)= & 1.6506 \\
 & & \log\tan\frac{1}{2}(A+B)= & \underline{0.1664} \\
\log a= & 1.6877 & \log\tan\frac{1}{2}(A-B)= & 0.0952 \\
\log\sin C= & 9.9689 & \frac{1}{2}(A-A)= & 51^\circ\ 14' \\
\log\csc A= & \underline{0.0103} & \frac{1}{2}(A+B)= & \underline{55^\circ\ 43'} \\
\log c= & 1.6669 & A= & 106^\circ\ 57' \\
c= & 46.44\ (v.\ \text{note}). & B= & 4^\circ\ 29'
\end{array}$$

Example II. Given $a=7.942$, $b=9.998$, $C=72^\circ\ 9'$.

$$A+B=180^\circ-C=107^\circ\ 51',$$

$$\tfrac{1}{2}(A+B)=53^\circ\ 55'.5.$$

NOTE. — c might have been obtained directly from the given parts by the formula $c^2=a^2+b^2-2bc \cos C$; but the above method is to be preferred, since the formula there used is adapted to logarithmic computation.

$$\begin{array}{rr}
a+b= \;\;17.942 & \text{colog}\,(a+b)=8.7461 \\
a-b=-2.056 & \log\,(a-b)=0.3131\,n \\
\log a=0.8999 & \log\tan\tfrac{1}{2}\,(\text{A}+\text{B})=\underline{0.1376} \\
\log\sin \text{C}=9.9786 & \log\tan\tfrac{1}{2}\,(\text{A}-\text{B})=9.1968\,n \\
\log\csc \text{A}=\underline{0.1507} & \tfrac{1}{2}\,(\text{A}-\text{B})=-8^\circ\ 56'.8 \\
\log c=1.0292 & \tfrac{1}{2}\,(\text{A}+\text{B})=53^\circ\ 55'.5 \\
\text{C}=\;\;10.70 & \text{A}=44^\circ\ 58'.7 \\
 & \text{B}=62^\circ\ 52'.3
\end{array}$$

In solving a triangle like the above, negative quantities can be avoided, if use is made of the following form of [68], —

$$\frac{b+a}{b-a}=\frac{\tan\frac{1}{2}\,(\text{B}+\text{A})}{\tan\frac{1}{2}\,(\text{B}-\text{A})}.$$

Example III.

Given $a=6.239$, $b=2.348$, $\text{C}=110^\circ\ 32'$.

Answers. $\text{A}=52^\circ\ 10'$, $\text{B}=17^\circ\ 18'$, $c=7.395$.

§ **88.** Case IV. *Given the three sides a, b, and c.*

$$s=\tfrac{1}{2}\,(a+b+c).$$

We can use here either [72], [73], [74], or [81]; but [74] or [81] are to be preferred, since, by their use, accurate results can be obtained for all values of the angles sought (*v.* § 62).

We will first use [74] which is, —

$$\tan\tfrac{1}{2}\,\text{A}=+\sqrt{\frac{(s-b)(s-c)}{s\,(s-a)}},$$

$$\tan\tfrac{1}{2}\,\text{B}=+\sqrt{\frac{(s-c)(s-a)}{s\,(s-b)}},$$

$$\tan\tfrac{1}{2}\,\text{C}=+\sqrt{\frac{(s-a)(s-b)}{s\,(s-c)}}:$$

$$\therefore \log\tan\tfrac{1}{2}\mathrm{A} = \tfrac{1}{2}\left[\log(s-b) + \log(s-c) + \operatorname{colog} s + \operatorname{colog}(s-a)\right].$$

$$\log\tan\tfrac{1}{2}\mathrm{B} = \tfrac{1}{2}\left[\log(s-c) + \log(s-a) + \operatorname{colog} s + \operatorname{colog}(s-b)\right].$$

$$\log\tan\tfrac{1}{2}\mathrm{C} = \tfrac{1}{2}\left[\log(s-a) + \log(s-b) + \operatorname{colog} s + \operatorname{colog}(s-c)\right].$$

Any two angles having been found, it is evident that the third can be found from the equation,

$$\mathrm{A} + \mathrm{B} + \mathrm{C} = 180^\circ.$$

It is better, however, to find each angle independently, and then to use the above equation to test the accuracy of the work.

Example. Given $a = 0.0886$, $b = 0.1642$, $c = 0.1022$.

$$\begin{array}{rlcrl}
2s = & 0.3550 & & & \\
s = & 0.1775 & & \log s = & \bar{1}.2492 \\
s-a = & 0.0889 & & \log(s-a) = & \bar{2}.9489 \\
s-b = & 0.0133 & & \log(s-b) = & \bar{2}.1239 \\
s-c = & 0.0753 & & \log(s-c) = & \bar{2}.8768 \\
\log(s-b) = & \bar{2}.1239 & & \log(s-c) = & \bar{2}.8768 \\
\log(s-c) = & \bar{2}.8768 & & \log(s-a) = & \bar{2}.9489 \\
\operatorname{colog} s = & 10.7508 & & \operatorname{colog} s = & 10.7508 \\
\operatorname{colog}(s-a) = & 11.0511 & & \operatorname{colog}(s-b) = & 11.8761 \\
2 & \overline{)18.8026} & & 2 & \overline{)20.4526} \\
\log\tan\tfrac{1}{2}\mathrm{A} = & 9.4013 & & \log\tan\tfrac{1}{2}\mathrm{B} = & 0.2263 \\
\tfrac{1}{2}\mathrm{A} = & 14^\circ\ 9' & & \tfrac{1}{2}\mathrm{B} = & 59^\circ\ 18 \\
\mathrm{A} = & 28^\circ\ 18' & & \mathrm{B} = & 118^\circ\ 36'
\end{array}$$

$$
\begin{aligned}
\log(s-a) &= \bar{2}.9489\\
\log(s-b) &= \bar{2}.1239\\
\operatorname{colog} s &= 10.7508\\
\operatorname{colog}(s-c) &= 11.1232\\
2\,&\underline{\overline{|\,18.9468}}\\
\log\tan\tfrac{1}{2}\mathrm{C} &= \underline{|\,9.4734}\\
\tfrac{1}{2}\mathrm{C} &= 16^\circ\ 34'\\
\mathrm{C} &= 33^\circ\ 8'.
\end{aligned}
$$

Test. $\mathrm{A}+\mathrm{B}+\mathrm{C}=180^\circ\ 02'$ (*v.* note).

§ **89.** We will now solve the problem by [81], which is well adapted to logarithmic computation when all the angles are required.

By [79], —

$$
\begin{aligned}
\log\tan\tfrac{1}{2}\mathrm{A} &= \log r+\operatorname{colog}(s-a),\\
\log\tan\tfrac{1}{2}\mathrm{B} &= \log r+\operatorname{colog}(s-b),\\
\log\tan\tfrac{1}{2}\mathrm{C} &= \log r+\operatorname{colog}(s-c).
\end{aligned}
$$

By [81], —

$$\log r=\tfrac{1}{2}[\log(s-a)+\log(s-b)+\log(s-c)+\operatorname{colog} s].$$

Example I. Given $a=409$, $b=241$, $c=182$.

$$
\begin{array}{rlrl}
s= & 416 & \operatorname{colog} s= & 7.3809\\
s-a= & 7 & \log(s-a)= & 0.8451\\
s-b= & 175 & \log(s-b)= & 2.2430\\
s-c= & 234 & \log(s-c)= & 2.3692\\
 & & 2 & \overline{|\,2.8382}\\
 & & \log r= & 1.4191\\
\log\tan\tfrac{1}{2}\mathrm{A}= & 0.5740 & \tfrac{1}{2}\mathrm{A}= & 75^\circ\ 4'\\
\log\tan\tfrac{1}{2}\mathrm{B}= & 9.1761 & \tfrac{1}{2}\mathrm{B}= & 8^\circ\ 32'\\
\log\tan\tfrac{1}{2}\mathrm{C}= & 9.0499 & \tfrac{1}{2}\mathrm{C}= & 6^\circ\ 24'
\end{array}
$$

Note.—If the error (*v.* § 62) does not exceed 2′, the work is probably correct.

$$A = 150° \, 8', \; B = 17° \, 4', \; C = 12° \, 48'.$$

TEST. $A + B + C = 180°$.

The formulas used in § § 88 and 89 were selected because they were adapted to logarithmic computation. It is *possible*, however, to solve the problem by [69].

Example II.

Given $a = 17.865$, $b = 13.349$, $c = 11.111$.

Answers.

$$A = 93° \, 20', \; B = 48° \, 16', \; C = 38° \, 24'.$$

EXAMPLES.

1. Given $A = 37° \, 12'$ $B = 123° \, 7'$ $a = .0841$ solve.
2. “ $A = 18° \, 17'$ $B = 94° \, 9'$ $b = 8400$ “
3. “ $B = 160° \, 19'$ $C = 10° \, 0'$ $b = 9.006$ “
4. “ $B = 85.18$ $C = 4° \, 42'$ $c = 1$ “
5. “ $a = 583.1$ $b = 503.7$ $A = 78° \, 13'$ “
6. “ $a = .0008$ $b = .0007$ $B = 40° \, 16'$ “
7. “ $a = 18.91$ $b = 9.846$ $A = 9° \, 19'$ “
8. “ $b = 1781$ $c = 982.7$ $B = 123° \, 16'$ “
9. “ $b = 81.04$ $c = 98.76$ $B = 24° \, 18'$ “
10. “ $a = 78.91$ $b = 68.73$ $C = 98° \, 17'$ “
11. “ $c = 16.70$ $a = 14.39$ $B = 16° \, 16'$ “
12. “ $a = 17.28$ $b = 9.861$ $c = 23.01$ “
13. “ $a = 8.067$ $b = 1.76$ $c = 7.001$ “
14. “ $a = 8.981$ $b = 10.08$ $B = 120° \, 8'$ “

15. Consider each of the following triangles, and determine whether it is possible, or impossible; if possible, whether it has one, two, or more solutions. Give in full the reasons for your conclusions.

(1.) $a=81^\circ\ 59'\ 32''$ $b=44^\circ\ 33'\ 12''$ $c=53^\circ\ 27'\ 16''$.
(2.) $a=38^\circ\ 36'$ $b=56^\circ\ 27'$ $c=84^\circ\ 12'$.
(3.) $a=121$ $b=115.36$ $b=94^\circ\ 15'$.
(4.) $a=30^\circ$ $b=175.2$ $a=150$.
(5.) $a=30^\circ$ $b=175.2$ $a=87.6$.
(6.) $a=34.27$ $b=27.34$ $c=62.18$.

16. Each of two ships, half a mile apart, finds the angles subtended by the other ship and a fort to be respectively 85° 15′ and 83° 45′: find the distance of each from the fort.

17. From a station, B, at the base of a mountain, its summit A is seen at an elevation of 60°; after walking one mile towards the summit, up a plane making an angle of 30° with the horizon, to another station, C, the angle BCA is observed to be 135°: find the height of the mountain in yards.

18. From a window in the same horizontal plane with the bottom of a steeple, the angle of elevation of the top of the steeple is 40°; and from another window, 18 feet directly above the former, the angle of elevation is 37° 30′: find the height and distance of the steeple.

19. Two men standing at the same point, C, observe the horizontal angle subtended by the line joining two inaccessible objects, A and B; they then move away, one in the direction AC to D, the other in the direction BC to E, until each observes the horizontal angle to be half what it was before: determine the distance AB when $\text{ACB}=30^\circ$, $\text{CD}=100$, $\text{CE}=200$.

20. To determine the distance between two inaccessible objects, A and B, a person measures the distance between any two points, C and D, in the same plane with A and B: at C he observes the angles DCA and ACB; and at D, the angles ADB and BDC: determine the distance AB when $\text{CD}=100$ feet, $\text{DCA}=40^\circ$, $\text{ACB}=30^\circ$, $\text{BDC}=20^\circ$.

21. Given B, a, and the area: solve the triangle.

22. The length of a road in which the ascent is 1 foot in 5, from the foot of the hill to the top is a mile and two-thirds. What will be the length of a zigzag road in which the ascent is 1 foot in 12?

23. Given $a = 24$, $b = 30$, $c = 18$: find the area.

24. Two angles of a triangle are 10° and 45°, and the included side is 10 feet: find the area.

25. Two sides of a triangle are equal to 3 and 12, and the included angle is 30°: find the hypothenuse of an equivalent right isosceles triangle.

26. The diagonals of a quadrilateral are in length a and b, and intersect at an angle A: prove area $= \frac{1}{2} a\, b \sin A$.

27. Having measured a base line of 400 yards, whose upper end was 24 feet higher than the lower one, in the same vertical plane with the top of a hill, I found the angles of elevation of the top of the hill from the lower and upper ends of the base line to be $5^\circ\ 17'$ and $3^\circ\ 17'$ respectively: find the height of the hill.

28. Being at sea, we saw two headlands, of which one bore S.W. by W., and the other W. by N. The chart showed that the first headland bore S.E. from the second, and was distant from it 23.25 miles. Find our distances from both headlands. *Ans.* 18.26 miles; 32.25 miles.

29. Two ships sail from the same port, the one S.W. 30 miles, and the other S.E. by S. 40 miles. Find the bearing and distance of the second ship from the first.

Ans. S. $74^\circ\ 30'$ W.; 45.08 miles.

30. An observer from a ship saw two headlands; the first bearing N.E. by E., and the second N.W. After he had sailed N.N.W. 10.25 miles, the first headland bore E. by N., and the second W.N.W. Find the bearing and distance of the first headland from the second.

Ans. N. $88^\circ\ 02'$ E., or nearly due east; 35.25 miles.

31. An observer saw two headlands; the first bearing S.E., the second E.S.E. After sailing E. by N. 10 miles, he saw the first bearing S. by E., and the second S.E. by S. Find the bearing and distance of the first headland from the second. *Ans.* S.S.W. 6.89 miles.

32. At one station the bearing of a cloud is N.N.W., and its angle of elevation 50° 35′. At a second station, bearing from the first N. by E., and distant 1 mile, the bearing of the cloud is W. by N. Find the height of the cloud, and its distance from each station.

Ans. 7727 feet; 10002 feet; 8838.

33. In the midst of a level plain, which is crossed by a straight road, stands a tower 250 feet high. An observer at the top of the tower sees an object which moves on the road. At first it bears N.N.W., and its angle of depression is 16° 08′; five minutes later it bears E. by S., and its angle of depression is 32° 18′. Find the direction of the road, its distance from the tower, and the rate at which the object is moving.

Ans. S.E. $\frac{1}{2}$ S.; 250.9 feet; 2.575 miles per hour.

34. Given: $a = 12.34$ chains, $b = 17.97$ chains, $C = 135°.04$.

To be computed: Area = 7.832 acres = 7 acres, 3 roods, 13 rods.

35. Given: $a = 17.95$ chains, $B = 100°$, $C = 70°$.

To be computed: Area = 85.90 acres = 85 acres, 3 roods, 24 rods.

36. Given: $a = 45.56$ chains, $b = 52.98$ chains, $c = 61.22$ chains.

To be computed: Area = 117.3 acres = 117 acres, 1.2 roods.

37. Given: $a = 32.56$ chains, $b = 57.84$ chains, $c = 44.44$ chains.

To be computed: Area = 71.93 acres = 71 acres, 3 roods, 29 rods.

38. Prove that the results contained in [85] § 85 are correct by the aid of the following principles: —

$$\sin B = \frac{b \sin A}{a}, \quad A + B + C = 180^\circ,$$

and, in any triangle, the greater side is opposite the greater angle.

APPENDIX I.

§ 1. The triangle of reference can be formed as stated in § 22, and also by dropping a perpendicular from any point of the terminal line on the axis Y′Y. Thus in Fig. 2_A (§ 4, Ap.), we may take OC′B′ as a triangle of reference for XOP′, and we then have $x = \text{C}'\text{B}'$, $y = \text{OC}'$, $r = \text{OB}'$; so that x is negative, and y and r positive. So for XOP‴, in the same figure, we may take $x = \text{C}'''\text{B}'''$, $y = \text{OC}'''$ $r = \text{OB}'''$.

§ 2. *To find the functions of* $\varphi \pm k\,360°$, *where* k *is any integer.*

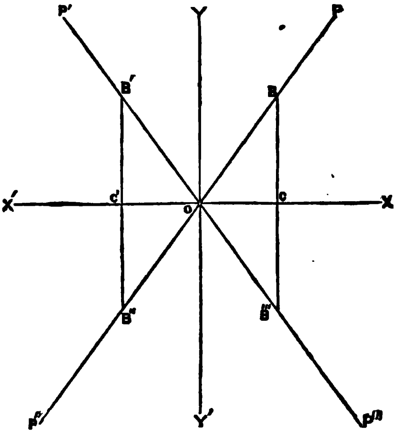
Let φ e any value of an angle XOP in any quadrant (Fig. 2_A, § 4, Ap.). If any multiple of 360° be added to or subtracted from φ, the result is also a value of the angle XOP (*v.* § 18); and the triangle of reference for φ is also a triangle of reference for $\varphi \pm k\,360°$. Hence the values of all the trigonometric functions are the same, both in numerical value and in sign, for φ and $\varphi \pm k\,360°$, that is, for *any value* of an angle XOP.

This property of the trigonometric functions is sometimes expressed by saying that they are *periodic* functions, admitting the period 360°, or 2π. We shall see presently, that the tangent and cotangent admit the simpler period, 180°, or π.

§ **3.** *To find the functions of* $-\varphi$.

Let φ be any value of an angle in any quadrant, as XOP, XOP′, XOP″, or XOP‴ (Fig. 1_A). Then will $-\varphi$ be a value of the angle XOP‴, XOP″, XOP′, or XOP. From any point of the terminal line of φ, as B, B′, B″, or B‴, drop a perpendicular on the initial line, and produce it to meet the terminal line of $-\varphi$ at B‴, B″, B′, or B. We thus form triangles of reference for φ and $-\varphi$; and, in the two triangles thus formed, the bases and hypothenuses are equal in numerical value and in sign, while the perpendiculars are numerically equal, but opposite in sign. Hence all the trigonometric functions of φ and $-\varphi$ are numerically equal; but those functions into which the perpendicular enters are opposite in sign. That is,

FIG. 1_A.

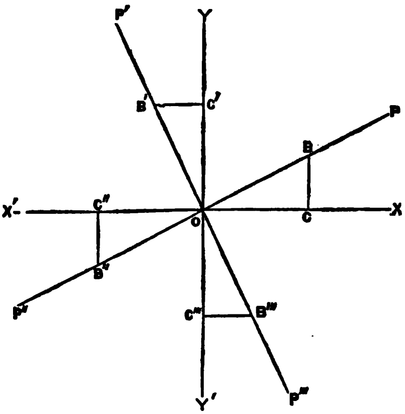


$$\left.\begin{array}{llll} \sin(-\varphi) = -\sin\varphi, & \csc(-\varphi) = -\csc\varphi, \\ \cos(-\varphi) = \cos\varphi, & \sec(-\varphi) = \sec\varphi, \\ \tan(-\varphi) = -\tan\varphi, & \operatorname{ctn}(-\varphi) = -\operatorname{ctn}\varphi. \end{array}\right\} \quad [1_A]$$

§ **4.** *To find the functions of* $90° + \varphi$.

Let φ be any value of an angle in any quadrant, as XOP, XOP′, XOP″, or XOP‴, in Fig. 2_A, where POP′, P′OP″, P″OP‴, and P‴OP are positive right angles. If 90° be added to any of these angles, it is converted into the next following angle of the series; and its triangle of reference is turned through a positive right angle (*v.* § 1, Ap.). Now, by virtue of this rotation, the base of the triangle of reference of φ becomes the perpendicular of the triangle of reference of $90° + \varphi$, and does not change its sign; but the perpendicular of the triangle of φ becomes the base of the triangle of $90° + \varphi$, and is reversed in sign; while the hypothenuse remains unchanged. That is, OC becomes OC′, C′B′ becomes C″B″, &c., CB becomes C′B, OC′ becomes OC″, &c ; or we may write —

FIG. 2A.



$$\begin{array}{l} \text{base} = \ \ \text{perp}' = -\text{base}'' = -\text{perp}''' = \text{base}^{\text{IV}}, \\ \text{perp} = -\text{base}' = -\text{perp}'' = \ \ \text{base}''' = \text{perp}^{\text{IV}}. \end{array}$$

Hence, for each value of φ, —

$$\sin\varphi = \frac{\text{perp}}{\text{hyp}} \text{ becomes } \frac{-\text{base}}{\text{hyp}} = -\cos(90° + \varphi);$$

$$\cos\varphi = \frac{\text{base}}{\text{hyp}} \text{ becomes } \frac{\text{perp}}{\text{hyp}} = \sin(90° + \varphi), \text{ \&c.}$$

Thus we have, —

$$\left.\begin{array}{ll} \sin(90°+\varphi) = \ \ \cos\varphi, & \csc(90°+\varphi) = \ \ \sec\varphi, \\ \cos(90°+\varphi) = -\sin\varphi, & \sec(90°+\varphi) = -\csc\varphi, \\ \tan(90°+\varphi) = -\text{ctn}\,\varphi, & \text{ctn}(90°+\varphi) = -\tan\varphi. \end{array}\right\} \quad [2_A]$$

§ 5. *To find the functions of* $90° - \varphi$ and $\varphi - 90°$.

Since formulas $[2_A]$ are true for *any* angle φ, we may substitute $-\varphi$ for φ in those formulas.

Thus we have, by $[1_A]$,

$$\left.\begin{array}{l}
\sin(90° - \varphi) = \cos(-\varphi) = \cos\varphi, \\
\cos(90° - \varphi) = -\sin(-\varphi) = \sin\varphi, \\
\tan(90° - \varphi) = -\operatorname{ctn}(-\varphi) = \operatorname{ctn}\varphi, \\
\operatorname{ctn}(90° - \varphi) = -\tan(-\varphi) = \tan\varphi, \\
\sec(90° - \varphi) = -\csc(-\varphi) = \csc\varphi, \\
\csc(90° - \varphi) = \sec(-\varphi) = \sec\varphi.
\end{array}\right\} \qquad [3_A]$$

Again: putting $90° - \varphi$ for φ in $[1_A]$, or $\varphi - 90°$ for $-\varphi$, we have, —

$$\left.\begin{array}{l}
\sin(\varphi - 90°) = -\cos\varphi, \\
\cos(\varphi - 90°) = \sin\varphi, \\
\tan(\varphi - 90°) = -\operatorname{ctn}\varphi, \\
\operatorname{ctn}(\varphi - 90°) = -\tan\varphi, \\
\sec(\varphi - 90°) = \csc\varphi, \\
\csc(\varphi - 90°) = -\sec\varphi.
\end{array}\right\} \qquad [4_A]$$

The angles φ and $90° - \varphi$ are complements of each other: therefore, from $[3_A]$, we see that

The cos of any angle is its *complement's sin*,

The csc of any angle is its *complement's sec*,

The ctn of any angle is its *complement's tan*.

§ 6. *To find the functions of* $k\,90° \pm \varphi$, *when* k *is any integer, positive or negative.*

The solution of this problem requires only the *repeated* use of $[2_A]$, $[3_A]$, and $[4_A]$.

Thus, putting $90° + \varphi$ for φ, in [2_A], we have, —

$$\left.\begin{array}{lcl} \sin(180°+\varphi) = & \cos(90°+\varphi) = & -\sin\varphi, \\ \cos(180°+\varphi) = & -\sin(90°+\varphi) = & -\cos\varphi, \\ \tan(180°+\varphi) = & -\text{ctn}(90°+\varphi) = & \tan\varphi. \end{array}\right\} \quad [5_A]$$

and the reciprocals of these results are the csc, sec, and ctn of $(180° + \varphi)$. We see by these formulas, that the tangent and cotangent admit the period 180°, or π (*v.* § 2, Ap.).

Putting $180° + \varphi$ for φ, in [2_A], we have again, by [5_A], —

$$\left.\begin{array}{lcl} \sin(270°+\varphi) = & \cos(180°+\varphi) = & -\cos\varphi, \\ \cos(270°+\varphi) = & -\sin(180°+\varphi) = & \sin\varphi, \\ \tan(270°+\varphi) = & -\text{ctn}(180°+\varphi) = & -\text{ctn}\,\varphi. \end{array}\right\} \quad [6_A]$$

Putting $270° + \varphi$ for φ, in [2_A], we have the functions of $360° + \varphi$, which we have already shown, in § 2 Ap., to be equal to the functions of φ.

Putting $90° - \varphi$ for φ, in [2_A], we have, by [3_A], —

$$\left.\begin{array}{lcl} \sin(180°-\varphi) = & \cos(90°-\varphi) = & \sin\varphi, \\ \cos(180°-\varphi) = & -\sin(90°-\varphi) = & -\cos\varphi, \\ \tan(180°-\varphi) = & -\text{ctn}(90°-\varphi) = & -\tan\varphi. \end{array}\right\} \quad [7_A]$$

Putting $180° - \varphi$ for φ, in [2_A], we have by [7_A], —

$$\left.\begin{array}{lcl} \sin(270°-\varphi) = & \cos(180°-\varphi) = & -\cos\varphi, \\ \cos(270°-\varphi) = & -\sin(180°-\varphi) = & -\sin\varphi, \\ \tan(270°-\varphi) = & -\text{ctn}(180°-\varphi) = & \text{ctn}\,\varphi. \end{array}\right\} \quad [8_A]$$

Putting $270° - \varphi$ for φ, in [2_A], we have the functions of $360° - \varphi$, which, by § 2, Ap., are equal to the functions of $-\varphi$.

Putting $\varphi - 90°$ for φ, in [4_A], we have, —

$$\left.\begin{array}{lcl} \sin(\varphi-180°) = & -\cos(\varphi-90°) = & -\sin\varphi, \\ \cos(\varphi-180°) = & \sin(\varphi-90°) = & -\cos\varphi, \\ \tan(\varphi-180°) = & -\text{ctn}(\varphi-90°) = & \tan\varphi. \end{array}\right\} \quad [9_A]$$

Putting $\varphi - 180^\circ$ for φ, in $[4_A]$, we have, by $[9_A]$, —

$$\left.\begin{array}{rll} \sin(\varphi - 270^\circ) = -\cos(\varphi - 180^\circ) = & \cos\varphi, \\ \cos(\varphi - 270^\circ) = \sin(\varphi - 180^\circ) = & -\sin\varphi, \\ \tan(\varphi - 270^\circ) = -\operatorname{ctn}(\varphi - 180^\circ) = & -\operatorname{ctn}\varphi. \end{array}\right\} [10_A]$$

Putting $\varphi - 270^\circ$ for φ, in $[4_A]$, we have the functions of $\varphi - 360^\circ$, which, by § 2 Ap., are equal to those of φ.

The student should now read the remarks at the end of § 33.

APPENDIX II.

COLLECTION OF FORMULAS.

AND OTHER USEFUL MATTER, FOR EASY REFERENCE.

§ 6. Circ. meas. of angle $\theta = \frac{s}{r}$.

Degree Measure.

Circular Measure.

§ 9.

$$1^\circ = \frac{\pi}{180} = 0.01745329$$

$$1' = \frac{1^\circ}{60} = 0.00029089$$

$$1'' = \frac{1'}{60} = 0.00000485$$

§ 10.

$$360^\circ = 2\pi$$

$$180^\circ = \pi$$

$$90^\circ = \tfrac{1}{2}\pi$$

$$45^\circ = \tfrac{1}{4}\pi$$

§ 9.

$$57^\circ.2957795 = \text{angular unit}$$

$$x\,x\,57^\circ.296.. = x$$

$$x^\circ = x \times 0.01745...$$

§ 16. $\text{XOP} = \text{XOB} + \text{BOY} + \text{YOP}$.

§ 18. $\text{AOB} = \varphi \pm k\,360^\circ$.

§ 24.

			is called the	abbreviated into
$\frac{\text{CB}}{\text{OB}}$	$\frac{y}{r}$	$\frac{\text{perpendicular}}{\text{hypothenuse}}$	sine of φ	$\sin\varphi$
$\frac{\text{OC}}{\text{OB}}$	$\frac{x}{r}$	$\frac{\text{base}}{\text{hypothenuse}}$	cosine of φ	$\cos\varphi$
$\frac{\text{CB}}{\text{OC}}$	$\frac{y}{x}$	$\frac{\text{perpendicular}}{\text{base}}$	tangent of φ	$\tan\varphi$
$\frac{\text{OC}}{\text{CB}}$	$\frac{x}{y}$	$\frac{\text{base}}{\text{perpendicular}}$	cotangent of φ	$\text{ctn}\,\varphi$
$\frac{\text{OB}}{\text{OC}}$	$\frac{r}{x}$	$\frac{\text{hypothenuse}}{\text{base}}$	secant of φ	$\sec\varphi$
$\frac{\text{OB}}{\text{CB}}$	$\frac{r}{y}$	$\frac{\text{hypothenuse}}{\text{perpendicular}}$	cosecant of φ	$\csc\varphi$

In addition to the above

	is called the	abbreviated into
$1 - \cos\varphi$	versed sine of φ	$\text{vers}\,\varphi$

§ 25.

φ	sin	cos	tan	ctn	sec	csc
1st quad	+	+	+	+	+	+
2d quad	+	—	—	—	—	+
3d quad	—	—	+	+	—	—
4th quad	—	+	—	—	+	—

§ 26.

$$\left.\begin{array}{lll} \sin\varphi = \dfrac{1}{\csc\varphi}, & \cos\varphi = \dfrac{1}{\sec\varphi}, & \tan\varphi = \dfrac{1}{\text{ctn}\,\varphi}, \\ \csc\varphi = \dfrac{1}{\sin\varphi}, & \sec\varphi = \dfrac{1}{\cos\varphi}, & \text{ctn}\,\varphi = \dfrac{1}{\tan\varphi}, \end{array}\right\} \qquad [1]$$

§ 27.

$$\frac{\sin\varphi}{\cos\varphi} = \tan\varphi \qquad [2]$$

$$\frac{\cos\varphi}{\sin\varphi} = \text{ctn}\,\varphi \qquad [3]$$

§ 28.

$$\sin^2\varphi + \cos^2\varphi = 1 \qquad [4]$$

$$\sec^2\varphi = 1 + \tan^2\varphi \qquad [5]$$

$$\csc^2\varphi = 1 + \text{ctn}^2\varphi \qquad [6]$$

§ 30. $\sin \varphi = \frac{1}{2} \text{ chord } 2\varphi$ [7]

§ 31.
$$\left\{\begin{array}{ll}
\left.\begin{array}{l}\sin(\varphi \pm k\,360^\circ) = \sin\varphi \\ \cos(\varphi \pm k\,360^\circ) = \cos\varphi\end{array}\right\} & [8] \\
\left.\begin{array}{l}\sin(180^\circ - \varphi) = \sin\varphi \\ \cos(180^\circ - \varphi) = -\cos\varphi\end{array}\right\} & [9] \\
\left.\begin{array}{l}\sin(180^\circ + \varphi) = -\sin\varphi \\ \cos(180^\circ + \varphi) = -\cos\varphi\end{array}\right\} & [10] \\
\left.\begin{array}{l}\sin(360^\circ - \varphi) = \sin(-\varphi) = -\sin\varphi \\ \cos(360^\circ - \varphi) = \cos(-\varphi) = \cos\varphi\end{array}\right\} & [11]
\end{array}\right.$$

§ 32.
$$\left\{\begin{array}{ll}
\left.\begin{array}{l}\sin(90^\circ - \varphi) = \cos\varphi \\ \cos(90^\circ - \varphi) = \sin\varphi\end{array}\right\} & [12]
\end{array}\right.$$

§ 33.
$$\left\{\begin{array}{ll}
\left.\begin{array}{l}\csc(90^\circ - \varphi) = \sec\varphi \\ \sec(90^\circ - \varphi) = \csc\varphi \\ \tan(90^\circ - \varphi) = \operatorname{ctn}\varphi \\ \operatorname{ctn}(90^\circ - \varphi) = \tan\varphi\end{array}\right\} & [12_1]
\end{array}\right.$$

§ 32.
$$\left\{\begin{array}{ll}
\left.\begin{array}{l}\sin(90^\circ + \varphi) = \cos\varphi \\ \cos(90^\circ + \varphi) = -\sin\varphi\end{array}\right\} & [13] \\
\left.\begin{array}{l}\sin(270^\circ - \varphi) = -\cos\varphi \\ \cos(270^\circ - \varphi) = -\sin\varphi\end{array}\right\} & [14] \\
\left.\begin{array}{l}\sin(270^\circ + \varphi) = -\cos\varphi \\ \cos(270^\circ + \varphi) = \sin\varphi\end{array}\right\} & [15]
\end{array}\right.$$

§ 33. Formulas 8–15 are easily remembered, if we note that when φ is coupled with 0°, 180°, or k 360°, the functions of the angles thus formed, and those of φ, are *numerically the same;* while when φ is coupled with 90°, or 270°, the functions of the angles thus formed, and those of φ, are *numerically complementary.*

The *algebraic sign* is determined in each case by supposing φ to be acute, and then noticing to what quadrant the resulting angle belongs.

§ 43.

Angle	sin	cos	tan	ctn	sec	csc
0°	0	1	0	∞	1	∞
30°	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$\sqrt{\frac{1}{3}}$	$\sqrt{3}$	$2\sqrt{\frac{1}{3}}$	2
45°	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$\sqrt{3}$	$\sqrt{\frac{1}{3}}$	2	$2\sqrt{\frac{1}{3}}$
90°	1	0	∞	0	∞	1
120°	$\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$	$-\sqrt{3}$	$-\sqrt{\frac{1}{3}}$	-2	$2\sqrt{\frac{1}{3}}$
135°	$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$	-1	-1	$-\sqrt{2}$	$\sqrt{2}$
150°	$\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	$-\sqrt{\frac{1}{3}}$	$-\sqrt{3}$	$-2\sqrt{\frac{1}{3}}$	2
180°	0	-1	0	∞	-1	∞
270°	-1	0	∞	0	∞	-1
360°	0	1	0	∞	1	∞

[16]

§ 44.

	sin.	cos.	tan.	ctn.	sec.	csc.
0°	∓ 0	$+1$	∓ 0	$\mp\infty$	$+1$	$\mp\infty$
1st qu.	inc.	dec.	inc.	dec.	inc.	dec.
90°	$+1$	± 0	$\pm\infty$	± 0	$\pm\infty$	$+1$
2d qu.	dec.	dec.	inc.	dec.	inc.	inc.
180°	± 0	-1	∓ 0	$\mp\infty$	-1	$\pm\infty$
3d qu.	dec.	inc.	inc.	dec.	dec.	inc.
270°	-1	∓ 0	$\pm\infty$	± 0	$\mp\infty$	-1
4th qu.	inc.	inc.	inc.	dec.	dec.	dec.
360°	∓ 0	$+1$	∓ 0	$\mp\infty$	$+1$	$\mp\infty$

[17]

$$\S\,46.\quad\begin{cases}\sin(\alpha\pm\beta)=\sin\alpha\cos\beta\pm\cos\alpha\sin\beta & [18]\\ \cos(\alpha\pm\beta)=\cos\alpha\cos\beta\mp\sin\alpha\sin\beta & [19]\end{cases}$$

§ 48.

$$\left\{\begin{aligned}&\tan(\alpha\pm\beta)=\frac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\tan\beta} &&[20]\\ &\operatorname{ctn}(\alpha\pm\beta)=\frac{\operatorname{ctn}\alpha\operatorname{ctn}\beta\mp1}{\operatorname{ctn}\beta\mp\operatorname{ctn}\alpha} &&[21]\end{aligned}\right.$$

§ 49.

$$\left\{\begin{aligned}&\frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}=\frac{\tan\alpha+\tan\beta}{\tan\alpha-\tan\beta}=\frac{\operatorname{ctn}\beta+\operatorname{ctn}\alpha}{\operatorname{ctn}\beta-\operatorname{ctn}\alpha} &&[22]\\ &\frac{\cos(\alpha+\beta)}{\cos(\alpha-\beta)}=\frac{1-\tan\alpha\tan\beta}{1+\tan\alpha\tan\beta}=\frac{\operatorname{ctn}\beta-\tan\alpha}{\operatorname{ctn}\beta+\tan\alpha} &&[23]\\ &\left.\begin{aligned}\sin(\alpha+\beta)\sin(\alpha-\beta)&=\sin^2\alpha-\sin^2\beta\\ &=\cos^2\beta-\cos^2\alpha\end{aligned}\right\} &&[24]\\ &\left.\begin{aligned}\cos(\alpha+\beta)\cos(\alpha-\beta)&=\cos^2\alpha-\sin\ \ \beta\\ &=\cos^2\beta-\sin^2\alpha\end{aligned}\right\} &&[25]\end{aligned}\right.$$

§ 50.

$$\left\{\begin{aligned}&\sin2\alpha=2\sin\alpha\cos\alpha &&[26]\\ &\left.\begin{aligned}\cos2\alpha&=\cos^2\alpha-\sin^2\alpha &&\text{(I.)}\\ &=2\cos^2\alpha-1 &&\text{(II.)}\\ &=1-2\sin^2\alpha &&\text{(III.)}\end{aligned}\right\} &&[27]\\ &\tan2\alpha=\frac{2\tan\alpha}{1-\tan^2\alpha} &&[28]\\ &\operatorname{ctn}2\alpha=\frac{\operatorname{ctn}^2\alpha-1}{2\operatorname{ctn}\alpha} &&[29]\end{aligned}\right.$$

§ 51.

$$\left\{\begin{aligned}&\sin\frac{\alpha}{2}=\sqrt{\tfrac{1}{2}(1-\cos\alpha)} &&[30]\\ &\cos\frac{\alpha}{2}=\sqrt{\tfrac{1}{2}(1+\cos\alpha)} &&[31]\\ &\tan\frac{\alpha}{2}=\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} &&[32]\\ &\operatorname{ctn}\frac{\alpha}{2}=\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}} &&[33]\end{aligned}\right.$$

$$\S\,52.\ \left\{\begin{array}{lr}
\text{ctn}\,2\,\alpha=\tfrac{1}{2}(\text{ctn}\,\alpha-\tan\alpha) & [34]\\
\csc 2\,\alpha=\tfrac{1}{2}\,(\tan\alpha+\text{ctn}\,\alpha) & [35]\\
\tan\tfrac{1}{2}\,\alpha=\csc\alpha-\text{ctn}\,\alpha & [36]\\
\text{ctn}\,\tfrac{1}{2}\,\alpha=\csc\alpha+\text{ctn}\,\alpha=\dfrac{1}{\csc\alpha-\text{ctn}\,\alpha} & [37]\\
\tan\,(45^\circ+\tfrac{1}{2}\,\alpha)=\sec\alpha+\tan\alpha. & [38]\\
\left.\begin{array}{l}\tan\,(45^\circ-\tfrac{1}{2}\,\alpha)=\text{ctn}\,(45^\circ+\tfrac{1}{2}\,\alpha)\\
\qquad\qquad=\sec\alpha-\tan\alpha=\dfrac{1}{\sec\alpha+\tan\alpha}\end{array}\right\} & [39]
\end{array}\right.$$

$$\S\,53.\ \left\{\begin{array}{lr}
\sin\,(\alpha+\beta)+\sin\,(\alpha-\beta)=2\sin\alpha\cos\beta & [40]\\
\sin\,(\alpha+\beta)-\sin\,(\alpha-\beta)=2\cos\alpha\sin\beta & [41]\\
\cos\,(\alpha+\beta)+\cos\,(\alpha-\beta)=2\cos\alpha\cos\beta & [42]\\
\cos\,(\alpha-\beta)-\cos\,(\alpha+\beta)=2\sin\alpha\sin\beta & [43]\\
\sin\alpha+\sin\beta=2\sin\tfrac{1}{2}\,(\alpha+\beta)\cos\tfrac{1}{2}\,(\alpha-\beta) & [44]\\
\sin\alpha-\sin\beta=2\cos\tfrac{1}{2}\,(\alpha+\beta)\sin\tfrac{1}{2}\,(\alpha-\beta) & [45]\\
\cos\alpha+\cos\beta=2\cos\tfrac{1}{2}\,(\alpha+\beta)\cos\tfrac{1}{2}\,(\alpha-\beta) & [46]\\
\cos\beta-\cos\alpha=2\sin\tfrac{1}{2}\,(\alpha+\beta)\sin\tfrac{1}{2}\,(\alpha-\beta) & [47]\\
\dfrac{\sin\alpha+\sin\beta}{\sin\alpha-\beta}=\dfrac{\tan\frac{1}{2}\,(\alpha+\beta)}{\tan\frac{1}{2}\,(\alpha-\beta)} & [48]\\
\dfrac{\cos\alpha-\cos\beta}{\cos\alpha+\cos\beta}=-\tan\tfrac{1}{2}\,(\alpha+\beta)\tan\tfrac{1}{2}\,(\alpha-\beta) & [49]\\
\dfrac{\sin\alpha\pm\sin\beta}{\cos\alpha+\cos\beta}=\quad\tan\tfrac{1}{2}\,(\alpha\pm\beta) & [50]\\
\dfrac{\sin\alpha\mp\sin\beta}{\cos\beta-\cos\alpha}=\quad\text{ctn}\,\tfrac{1}{2}\,(\alpha\pm\beta) & [51]
\end{array}\right.$$

$$\S\,54.\quad \sin\alpha<\alpha<\tan\alpha \qquad [52]$$

$$\S\,55.\quad \cos\alpha>1-\tfrac{1}{2}\,\alpha^2 \qquad [59]$$

$$\S\,56.\ \left\{\begin{array}{l}\sin\alpha>\alpha-\tfrac{1}{6}\,\alpha^3\\ \cos\alpha<1-\tfrac{1}{2}\,\alpha^2+\tfrac{1}{24}\,\alpha^4\end{array}\right. \qquad [63]$$

§ 65. $\left\{\begin{array}{l} b=\sqrt{(c+a)\ (c-a)} \\ a=\sqrt{(c+b)\ (c-b)} \end{array}\right\}$ [65]

§ 68. $\sin\frac{1}{2}\mathrm{A}=\sqrt{\frac{c-b}{2c}},\ \sin\frac{1}{2}\mathrm{B}=\sqrt{\frac{c-a}{2c}}$ [66]

§ 71. $\frac{a}{\sin\mathrm{A}}=\frac{b}{\sin\mathrm{B}}=\frac{c}{\sin\mathrm{C}}$ [67]

§ 72. $\frac{a+b}{a-b}=\frac{\tan\frac{1}{2}(\mathrm{A}+\mathrm{B})}{\tan\frac{1}{2}(\mathrm{A}-\mathrm{B})}$ [68]

§ 73. $\left\{\begin{array}{ll} * \ a^2=b^2+c^2-2bc\cos\mathrm{A} & [69] \\ d^2=m^2+n^2+2mn\cos w & [70] \end{array}\right.$

§ 74. $* \ a=b\cos\mathrm{C}+c\cos\mathrm{B}$ [71]

§ 75. $\left\{\begin{array}{ll} * \ \sin\frac{1}{2}\mathrm{A}=+\sqrt{\frac{(s-b)(s-c)}{bc}} & [72] \\ * \ \cos\frac{1}{2}\mathrm{A}=+\sqrt{\frac{s(s-a)}{bc}} & [73] \end{array}\right.$

§ 76. $* \ \tan\frac{1}{2}\mathrm{A}=+\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$ [74]

§ 77. $\left\{\begin{array}{ll} * \ \mathrm{K}=\frac{1}{2}bc\sin\mathrm{A} & [75] \\ * \ \mathrm{K}=\frac{a^2\sin\mathrm{B}\sin\mathrm{C}}{\sin(\mathrm{B}+\mathrm{C})} & [76] \\ \mathrm{K}=\sqrt{s(s-a)(s-b)(s-c)} & [77] \end{array}\right.$

§ 78. $\mathrm{K}=sr$ [78]

§ 79. $\left\{\begin{array}{ll} r=\frac{\mathrm{K}}{s}=\sqrt{\frac{(s-a)(s-b)(s-c)}{s}} & [79] \\ r=s\tan\frac{1}{2}\mathrm{A}\tan\frac{1}{2}\mathrm{B}\tan\frac{1}{2}\mathrm{C} & [80] \end{array}\right.$

§ 80. $* \ \tan\frac{1}{2}\mathrm{A}=\frac{r}{s-a}$ [81]